Exercise 25

Solve the diffusion problem with a source function $q(x,t)$

$$u_t = \kappa u_{xx} + q(x,t), \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = 0 \quad \text{for } -\infty < x < \infty.$$

Show that the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa}} \int_0^t (t-\tau)^{-\frac{1}{2}}\, d\tau \int_{-\infty}^{\infty} q(k,\tau) \exp\left[-\frac{(x-k)^2}{4\kappa(t-\tau)}\right] dk.$$

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_t\} = \mathcal{F}\{\kappa u_{xx} + q(x,t)\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_t\} = \kappa\mathcal{F}\{u_{xx}\} + \mathcal{F}\{q(x,t)\}$$

Transform the derivatives with the relations above.

$$\frac{dU}{dt} = \kappa(ik)^2 U + Q(k,t)$$

Expand the coefficient of U and bring the term to the left side.

$$\frac{dU}{dt} + \kappa k^2 U = Q(k,t) \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial condition as well. Taking the Fourier transform of the initial condition gives

$$u(x,0) = 0 \qquad \rightarrow \qquad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{0\}$$
$$U(k,0) = 0. \tag{2}$$

Equation (1) is an ODE in t, so k is treated as a constant. It is first-order and inhomogeneous, so it can be solved with an integrating factor of the form,

$$I = e^{\int^t \kappa k^2\, ds} = e^{\kappa k^2 t}.$$

Multiply both sides of equation (1) by I.

$$e^{\kappa k^2 t}\frac{dU}{dt} + \kappa k^2 e^{\kappa k^2 t} U = Q(k,t)e^{\kappa k^2 t}$$

The ODE is now exact, and the left side can be written as $d/dt(IU)$ as a result of the product rule.

$$\frac{d}{dt}(e^{\kappa k^2 t}U) = Q(k,t)e^{\kappa k^2 t}$$

Integrate both sides with respect to t.

$$e^{\kappa k^2 t}U = \int_0^t Q(k,\tau)e^{\kappa k^2 \tau}\,d\tau + C$$

The lower limit of integration 0 is arbitrary; the constant of integration C will be adjusted accordingly to satisfy the initial condition. Speaking of which, we can apply equation (2) now to determine C.

$$0 = C$$

Divide both sides by $e^{\kappa k^2 t}$ to solve for U.

$$U(k,t) = e^{-\kappa k^2 t}\int_0^t Q(k,\tau)e^{\kappa k^2 \tau}\,d\tau$$

Bring the exponential inside the integral and combine it with the one in the integrand.

$$U(k,t) = \int_0^t Q(k,\tau)e^{\kappa k^2(\tau - t)}\,d\tau$$

Now that $U(k,t)$ is solved for, we can obtain $u(x,t)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\{U(k,t)\} \\ &= \mathcal{F}^{-1}\left\{\int_0^t Q(k,\tau)e^{\kappa k^2(\tau-t)}\,d\tau\right\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, $Q(k,\tau)$ and $e^{\kappa k^2(\tau-t)}$, we can apply the convolution theorem, which states that

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

Looking up the Fourier transform of e^{-ax^2} in a table,

$$\mathcal{F}\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}},$$

and comparing it with $U(k,t)$, we want a to be chosen so that

$$-\frac{1}{4a} = \kappa(\tau - t) \quad \rightarrow \quad a = \frac{1}{4\kappa(t-\tau)}.$$

Hence, after taking the inverse Fourier transform of both sides and multiplying both sides by $\sqrt{2a}$,

$$\sqrt{2a}e^{-ax^2} = \mathcal{F}^{-1}\left\{e^{-\frac{k^2}{4a}}\right\}.$$

Plugging in the value of a we found, we now know the inverse Fourier transform of the exponential function and can apply the convolution theorem.

$$\frac{1}{\sqrt{2\kappa(t-\tau)}}e^{-\frac{x^2}{4\kappa(t-\tau)}} = \mathcal{F}^{-1}\left\{e^{\kappa k^2(\tau-t)}\right\}$$

By the convolution theorem,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^t q(\xi,\tau)\frac{1}{\sqrt{2\kappa(t-\tau)}}e^{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}}\,d\tau\,d\xi.$$

Switch the order of the integrals and move the constants out in front.

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa}}\int_0^t \frac{1}{\sqrt{t-\tau}}\left[\int_{-\infty}^{\infty} q(\xi,\tau)e^{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}}\,d\xi\right]d\tau$$

Change ξ to k. Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_{-\infty}^{\infty} q(k,\tau)e^{-\frac{(x-k)^2}{4\kappa(t-\tau)}}\,dk.$$