Exercise 27

Use the double Fourier transform to solve the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}, \quad -\infty < x, t < \infty,$$
$$u(0,t) = f(t), \quad u_x(0,t) = g(t), \quad \text{for } -\infty < t < \infty,$$

where a, b, c are constants and $f(t)$ and $g(t)$ are arbitrary functions of t.

Solution

I don't see how to apply the double Fourier transform, so I'll just solve it the way I know how. The PDE is defined for $-\infty < t < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here with respect to t as

$$\mathcal{F}_t\{u(x,t)\} = U(x,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} u(x,t)\, dt,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}_t\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{d^n U}{dx^n}$$
$$\mathcal{F}_t\left\{\frac{\partial^n u}{\partial t^n}\right\} = (ik)^n U(x,k)$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}_t\{u_{tt} + au_t + bu\} = \mathcal{F}_t\{c^2 u_{xx}\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_t\{u_{tt}\} + a\mathcal{F}_t\{u_t\} + b\mathcal{F}_t\{u\} = c^2 \mathcal{F}_t\{u_{xx}\}$$

Transform the derivatives with the relations above.

$$(ik)^2 U + a(ik)U + bU = c^2 \frac{d^2 U}{dx^2}$$

Factor U and divide both sides by c^2.

$$\frac{d^2 U}{dx^2} = \left(\frac{-k^2 + iak + b}{c^2}\right) U$$

Factor out -1 and let

$$\alpha = \alpha(k) = \sqrt{\frac{k^2 - iak - b}{c^2}}.$$

Then

$$\frac{d^2 U}{dx^2} = -\alpha^2 U \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(0,t) = f(t) \quad \rightarrow \quad \mathcal{F}_t\{u(0,t)\} = \mathcal{F}_t\{f(t)\}$$
$$U(0,k) = F(k) \tag{2}$$
$$\frac{\partial u}{\partial x}(0,t) = g(t) \quad \rightarrow \quad \mathcal{F}_t\left\{\frac{\partial u}{\partial x}(0,t)\right\} = \mathcal{F}_t\{g(t)\}$$
$$\frac{dU}{dx}(0,k) = G(k). \tag{3}$$

Equation (1) is an ODE in x, so k is treated as a constant. The solution to the ODE is given in terms of sine and cosine.

$$U(x,k) = A(k)\cos\alpha x + B(k)\sin\alpha x$$

Apply the first initial condition, equation (2).

$$U(0,k) = A(k) = F(k)$$

In order to apply the second initial condition, differentiate $U(x,k)$ with respect to x.

$$\frac{dU}{dx}(x,k) = -\alpha A(k)\sin\alpha x + \alpha B(k)\cos\alpha x$$

Now apply equation (3).

$$\frac{dU}{dx}(0,k) = \alpha B(k) = G(k) \quad \rightarrow \quad B(k) = \frac{G(k)}{\alpha}$$

Thus, the solution to the ODE that satisfies the initial conditions is

$$U(x,k) = F(k)\cos\alpha x + \frac{G(k)}{\alpha}\sin\alpha x$$

In order to change back to $u(x,t)$, we have to take the inverse Fourier transform of $U(x,k)$. It is defined as

$$\mathcal{F}^{-1}\{U(x,k)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(x,k)e^{ikt}\,dk.$$

Plugging $U(x,k)$ into the definition, we therefore have

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(k)\cos\alpha x + \frac{G(k)}{\alpha}\sin\alpha x\right]e^{ikt}\,dk,$$

where

$$\alpha = \alpha(k) = \sqrt{\frac{k^2 - iak - b}{c^2}}$$
$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ikt}\,dt$$
$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-ikt}\,dt.$$