Exercise 31

Solve the telegraph equation in Exercise 29 with $V(x,0) = 0 = V_t(x,0)$ for the Heaviside distortionless cable ($\frac{R}{L} = \frac{G}{C} = \text{const.} = k$) with the boundary data $V(0,t) = V_0 f(t)$ and $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$, where V_0 is constant and $f(t)$ is an arbitrary function of t. Explain the physical significance of the solution.

Solution

The telegraph equation in Exercise 29 is

$$LCV_{tt} - V_{xx} + (LG + RC)V_t + RGV = 0.$$

Divide both sides by LC.

$$V_{tt} - \frac{1}{LC}V_{xx} + \left(\frac{G}{C} + \frac{R}{L}\right)V_t + \frac{RG}{LC}V = 0$$

Since

$$\frac{R}{L} = \frac{G}{C} = k \quad \text{and} \quad c^2 = \frac{1}{LC},$$

the equation simplifies to

$$V_{tt} - c^2 V_{xx} + 2kV_t + k^2 V = 0.$$

Because we're given two initial conditions and $t > 0$, this PDE can be solved using the Laplace transform. It is defined as

$$\mathcal{L}\{V(x,t)\} = \overline{V}(x,s) = \int_0^t e^{-st} V(x,t)\,dt,$$

which means the derivatives of V with respect to x and t transform as follows.

$$\mathcal{L}\left\{\frac{\partial^n V}{\partial x^n}\right\} = \frac{d^n \overline{V}}{dx^n}$$

$$\mathcal{L}\left\{\frac{\partial V}{\partial t}\right\} = s\overline{V}(x,s) - V(x,0)$$

$$\mathcal{L}\left\{\frac{\partial^2 V}{\partial t^2}\right\} = s^2\overline{V}(x,s) - sV(x,0) - V_t(x,0)$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{V_{tt} - c^2 V_{xx} + 2kV_t + k^2 V\} = \mathcal{L}\{0\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{V_{tt}\} - c^2\mathcal{L}\{V_{xx}\} + 2k\mathcal{L}\{V_t\} + k^2\mathcal{L}\{V\} = 0$$

Transform the derivatives with the relations above.

$$s^2\overline{V} - sV(x,0) - V_t(x,0) - c^2\frac{d^2\overline{V}}{dx^2} + 2k[s\overline{V} - V(x,0)] + k^2\overline{V} = 0$$

Plug in the initial conditions, $V(x,0)=0$ and $V_t(x,0)=0$, and factor $\overline{V}$.

$$c^2\frac{d^2\overline{V}}{dx^2}=(s^2+2ks+k^2)\overline{V}$$

Divide both sides by c^2 and recognize that the term multiplying $\overline{V}$ is a perfect square.

$$\frac{d^2\overline{V}}{dx^2}=\frac{(s+k)^2}{c^2}\overline{V}$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\overline{V}(x,s)=A(s)e^{\frac{s+k}{c}x}+B(s)e^{-\frac{s+k}{c}x}$$

In order to satisfy the condition that $V(x,t)\to 0$ as $x\to\infty$, we require that $A(s)=0$.

$$\overline{V}(x,s)=B(s)e^{-\frac{s+k}{c}x}$$

To determine $B(s)$, we have to use the boundary condition at $x=0$, $V(0,t)=V_0f(t)$. Take the Laplace transform of both sides of it.

$$\begin{aligned}\mathcal{L}\{V(0,t)\}&=\mathcal{L}\{V_0f(t)\}\\ \overline{V}(0,s)&=V_0F(s)\end{aligned}$$

Plug in $x=0$ into the formula for $\overline{V}$ and use the boundary condition.

$$\overline{V}(0,s)=B(s)=V_0F(s)$$

Thus,

$$\overline{V}(x,s)=V_0F(s)e^{-\frac{s+k}{c}x}.$$

Now that we have $\overline{V}(x,s)$ we can obtain $V(x,t)$ by taking the inverse Laplace transform of it.

$$\begin{aligned}V(x,t)=\mathcal{L}^{-1}\{\overline{V}(x,s)\}&=\mathcal{L}^{-1}\left\{V_0F(s)e^{-\frac{s+k}{c}x}\right\}\\ &=\mathcal{L}^{-1}\left\{V_0F(s)e^{-\frac{s}{c}x}e^{-\frac{k}{c}x}\right\}\\ &=V_0e^{-\frac{k}{c}x}\mathcal{L}^{-1}\left\{F(s)e^{-\frac{s}{c}x}\right\}\end{aligned}$$

Here we make use of the fact that

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}=H(t-a)f(t-a).$$

Therefore,

$$V(x,t)=V_0e^{-\frac{k}{c}x}f\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right),$$

where

$$c^2=\frac{1}{LC}.$$