Exercise 32

Solve the inhomogeneous partial differential equation

$$u_{xt} = -\omega \sin \omega t, \quad t > 0,$$
$$u(x,0) = x, \qquad u(0,t) = 0.$$

Solution

Solution by Partial Integration

Integrate both sides of the PDE partially with respect to t to get rid of the t-derivative on u.

$$\int^t u_{xt}\big|_{t=s}\, ds = \int^t -\omega \sin \omega s \, ds + f(x),$$

where f is an arbitrary function.

$$u_x = \cos \omega t + f(x)$$

To eliminate the x-derivative, integrate both sides partially with respect to x.

$$\int^x u_x\big|_{x=r}\, dr = \int^x [\cos \omega t + f(r)]\, dr + g(t),$$

where g is another arbitrary function.

$$u(x,t) = x \cos \omega t + F(x) + g(t),$$

where F is another arbitrary function. To determine them, we use the provided initial and boundary conditions.

$$u(x,0) = x + F(x) + g(0) = x \quad \rightarrow \quad F(x) + g(0) = 0$$
$$u(0,t) = F(0) + g(t) = 0$$

Set $F(x) = 0$ and $g(t) = 0$ to satisfy the conditions. Therefore,

$$u(x,t) = x \cos \omega t.$$

Solution by the Laplace Transform

Since we're given an initial condition and $t > 0$, we can solve this PDE with the Laplace transform. It is defined as

$$\mathcal{L}\{u(x,t)\} = \bar{u}(x,s) = \int_0^t e^{-st} u(x,t)\, dt,$$

which means the derivatives of u with respect to x and t transform as follows.

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{d^n \bar{u}}{dx^n}$$
$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0)$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{u_{xt}\} = \mathcal{L}\{-\omega \sin \omega t\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{u_{xt}\} = -\omega\mathcal{L}\{\sin \omega t\}$$

Transform the derivative on the left with the relations above.

$$\frac{d}{dx}[\bar{u}(x,s) - u(x,0)] = -\omega\frac{\omega}{s^2+\omega^2}$$

Substitute the initial condition, $u(x, 0) = x$.

$$\frac{d}{dx}[\bar{u}(x,s) - x] = -\frac{\omega^2}{s^2+\omega^2}$$

Evaluate the derivative on the left side.

$$\frac{d\bar{u}}{dx} - 1 = -\frac{\omega^2}{s^2+\omega^2}$$

Solve for $d\bar{u}/dx$.

$$\frac{d\bar{u}}{dx} = \frac{s}{s^2+\omega^2}$$

The PDE has thus been reduced to an ODE, which is first-order and can be solved by integrating both sides with respect to x.

$$\bar{u}(x,s) = \frac{s}{s^2+\omega^2}x + C$$

To determine the constant C, we use the boundary condition at $x = 0$, $u(0, t) = 0$. Take the Laplace transform of both sides of it.

$$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{0\}$$
$$\bar{u}(0,s) = 0$$

Plugging in $x = 0$ into the formula for $\bar{u}$ and using the boundary condition, we have

$$\bar{u}(0,s) = C = 0.$$

Thus,

$$\bar{u}(x,s) = \frac{s}{s^2+\omega^2}x.$$

All that's left to do now is to take the inverse Laplace transform of this to get $u(x, t)$.

$$\begin{aligned} u(x,t) = \mathcal{L}^{-1}\{\bar{u}(x,s)\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}x\right\} \\ &= x\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} \end{aligned}$$

Therefore,

$$u(x,t) = x\cos \omega t.$$