Exercise 33

Find the solution of the inhomogeneous equation

$$\frac{1}{c^2}u_{tt} - u_{xx} = k\sin\left(\frac{\pi x}{a}\right), \quad 0 < x < a,\ t > 0,$$
$$u(x,0) = 0 = u_t(x,0) \quad \text{for } 0 < x < a,$$
$$u(0,t) = 0 = u(a,t) \quad \text{for } t > 0.$$

Solution

This problem can be solved with the Laplace transform since we have initial conditions and $t > 0$. It is defined as

$$\mathcal{L}\{u(x,t)\} = \bar{u}(x,s) = \int_0^t e^{-st}u(x,t)\,dt,$$

which means the derivatives of u with respect to x and t transform as follows.

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial x^n}\right\} = \frac{d^n\bar{u}}{dx^n}$$
$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\bar{u}(x,s) - su(x,0) - u_t(x,0)$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\left\{\frac{1}{c^2}u_{tt} - u_{xx}\right\} = \mathcal{L}\left\{k\sin\frac{\pi x}{a}\right\}$$

The Laplace transform is a linear operator.

$$\frac{1}{c^2}\mathcal{L}\left\{u_{tt}\right\} - \mathcal{L}\{u_{xx}\} = k\sin\frac{\pi x}{a}\mathcal{L}\left\{1\right\}$$

Transform the derivatives with the relations above.

$$\frac{1}{c^2}[s^2\bar{u}(x,s) - su(x,0) - u_t(x,0)] - \frac{d^2\bar{u}}{dx^2} = \frac{k}{s}\sin\frac{\pi x}{a}$$

Substitute the initial conditions, $u(x,0) = 0$ and $u_t(x,0) = 0$.

$$\frac{s^2}{c^2}\bar{u} - \frac{d^2\bar{u}}{dx^2} = \frac{k}{s}\sin\frac{\pi x}{a}$$

Multiply both sides by -1.

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{k}{s}\sin\frac{\pi x}{a}$$

The PDE has thus been reduced to an ODE. This ODE is a second-order inhomogeneous equation, so the general solution for it is the sum of a complementary solution and a particular solution.

$$\bar{u} = \bar{u}_c + \bar{u}_p,$$

$\bar{u}_c$ is the solution to the associated homogeneous equation,

$$\frac{d^2\bar{u}_c}{dx^2} - \frac{s^2}{c^2}\bar{u}_c = 0,$$

which is

$$\bar{u}_c = A(s)\cosh\frac{sx}{c} + B(s)\sinh\frac{sx}{c}.$$

Because the inhomogeneous term is sine and only even derivatives are present on the left side, $\bar{u}_p$ has the form, $C_1 \sin\frac{\pi x}{a}$. Plugging this form into the ODE allows us to determine the constant C_1. We get

$$-\frac{C_1(c^2\pi^2 + a^2s^2)}{a^2c^2}\sin\frac{\pi x}{a} = -\frac{k}{s}\sin\frac{\pi x}{a} \quad \rightarrow \quad C_1 = \frac{ka^2c^2}{s(a^2s^2 + c^2\pi^2)}.$$

Thus, the general solution for $\bar{u}(x, s)$ is

$$\bar{u}(x, s) = A(s)\cosh\frac{sx}{c} + B(s)\sinh\frac{sx}{c} + \frac{ka^2c^2}{s(a^2s^2 + c^2\pi^2)}\sin\frac{\pi x}{a}.$$

Now we use the provided boundary conditions at $x = 0$ and $x = a$ to determine $A(s)$ and $B(s)$. Take the Laplace transform of both sides of them.

$$\begin{aligned} u(0, t) = 0 \quad &\rightarrow \quad \mathcal{L}\{u(0, t)\} = \mathcal{L}\{0\} \\ &\qquad\qquad \bar{u}(0, s) = 0 \qquad (1) \\ u(a, t) = 0 \quad &\rightarrow \quad \mathcal{L}\{u(a, t)\} = \mathcal{L}\{0\} \\ &\qquad\qquad \bar{u}(a, s) = 0 \qquad (2) \end{aligned}$$

Setting $x = 0$ and $x = a$ and using equations (1) and (2), we have

$$\begin{aligned} \bar{u}(0, s) &= A(s) = 0 \\ \bar{u}(a, s) &= B(s)\sinh\frac{sa}{c} = 0 \quad \rightarrow \quad B(s) = 0. \end{aligned}$$

Thus, the solution reduces to

$$\bar{u}(x, s) = \frac{ka^2c^2}{s(a^2s^2 + c^2\pi^2)}\sin\frac{\pi x}{a}.$$

Now that we have $\bar{u}(x, s)$, we can change back to $u(x, t)$ by taking the inverse Laplace transform of it. Before we do so, rewrite the solution in a more convenient form. Start by using partial fraction decomposition.

$$\bar{u}(x, s) = \left[\frac{ka^2}{\pi^2}\cdot\frac{1}{s} - \frac{ka^4 s}{\pi^2(a^2s^2 + c^2\pi^2)}\right]\sin\frac{\pi x}{a}$$

Factor out a^2 from the second denominator.

$$\bar{u}(x, s) = \left(\frac{ka^2}{\pi^2}\cdot\frac{1}{s} - \frac{ka^2}{\pi^2}\cdot\frac{s}{s^2 + \frac{c^2\pi^2}{a^2}}\right)\sin\frac{\pi x}{a}$$

Factor ka^2/π^2.

$$\bar{u}(x, s) = \frac{ka^2}{\pi^2}\left(\frac{1}{s} - \frac{s}{s^2 + \frac{c^2\pi^2}{a^2}}\right)\sin\frac{\pi x}{a}$$

Now take the inverse Laplace transform.

$$\begin{aligned}
u(x,t) = \mathcal{L}^{-1}\{\bar{u}(x,s)\} &= \mathcal{L}^{-1}\left\{\frac{ka^2}{\pi^2}\left(\frac{1}{s} - \frac{s}{s^2 + \frac{c^2\pi^2}{a^2}}\right)\sin\frac{\pi x}{a}\right\} \\
&= \frac{ka^2}{\pi^2}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2 + \frac{c^2\pi^2}{a^2}}\right)\right\}\sin\frac{\pi x}{a} \\
&= \frac{ka^2}{\pi^2}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \frac{c^2\pi^2}{a^2}}\right\}\right)\sin\frac{\pi x}{a} \\
&= \frac{ka^2}{\pi^2}\left(1 - \cos\frac{c\pi t}{a}\right)\sin\frac{\pi x}{a}
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{ka^2}{\pi^2}\left(1 - \cos\frac{c\pi t}{a}\right)\sin\frac{\pi x}{a}.$$

This answer is in disagreement with the answer at the back of the book. The factor in front there is $k/c^2\pi^2$. My answer satisfies the PDE and all initial and boundary conditions, but the book's answer does not.