Exercise 38

Solve the telegraph equation

$$u_{tt} - c^2 u_{xx} + 2au_t = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = g(x), \quad -\infty < x < \infty.$$

Solution

This exercise is the same as Exercise 1.12; α has been replaced with $2a$ and $f(x)$ is set to 0 here. The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{tt} - c^2 u_{xx} + 2au_t\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{tt}\} - c^2\mathcal{F}\{u_{xx}\} + 2a\mathcal{F}\{u_t\} = 0$$

Transform the derivatives with the relations above.

$$\frac{d^2U}{dt^2} - c^2(ik)^2 U + 2a\frac{dU}{dt} = 0$$

Expand the coefficient of U.

$$\frac{d^2U}{dt^2} + 2a\frac{dU}{dt} + c^2k^2U = 0 \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(x,0) = 0 \quad \rightarrow \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{0\}$$
$$U(k,0) = 0 \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \rightarrow \quad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x,0)\right\} = \mathcal{F}\{g(x)\}$$
$$\frac{dU}{dt}(k,0) = G(k). \tag{3}$$

Equation (1) is an ODE in t, so k is treated as a constant. We can solve it with the Laplace transform since $t > 0$. The Laplace transform of $U(k,t)$ is defined as

$$\mathcal{L}\{U(k,t)\} = \overline{U}(k,s) = \int_0^{\infty} e^{-st} U(k,t)\, dt,$$

so the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dU}{dt}\right\} = s\overline{U}(k,s) - U(k,0) \tag{4}$$

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} = s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0) \tag{5}$$

Take the Laplace transform of both sides of equation (1).

$$\mathcal{L}\left\{\frac{d^2U}{dt^2} + 2a\frac{dU}{dt} + c^2k^2U\right\} = \mathcal{L}\{0\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} + 2a\mathcal{L}\left\{\frac{dU}{dt}\right\} + c^2k^2\mathcal{L}\{U\} = 0$$

Use equations (4) and (5) here.

$$\left[s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0)\right] + 2a[s\overline{U}(k,s) - U(k,0)] + c^2k^2\overline{U}(k,s) = 0$$

Expand the left side and substitute equations (2) and (3).

$$s^2\overline{U}(k,s) - G(k) + 2as\overline{U}(k,s) + c^2k^2\overline{U}(k,s) = 0$$

The ODE has thus been reduced to an algebraic equation. Factor $\overline{U}(k,s)$ and bring the terms without it to the right side.

$$(s^2 + 2as + c^2k^2)\overline{U}(k,s) = G(k)$$

Divide both sides by $s^2 + 2as + c^2k^2$ to solve for $\overline{U}$.

$$\overline{U}(k,s) = \frac{G(k)}{s^2 + 2as + c^2k^2}.$$

In order to change back to $u(x,t)$, we have to take the inverse Laplace transform of $\overline{U}(k,s)$ to get $U(k,t)$ and then take the inverse Fourier transform of it. Our task now is to write $\overline{U}$ in a form that we can easily transform. The inverse Laplace transform we will eventually use is

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{at}\sin bt, \tag{6}$$

so we want to write $\overline{U}$ in this form. Complete the square in the denominator.

$$\overline{U}(k,s) = \frac{G(k)}{(s+a)^2 + (c^2k^2 - a^2)}$$

Multiply the numerator and denominator by $\sqrt{c^2k^2 - a^2}$.

$$\overline{U}(k,s) = \frac{G(k)}{\sqrt{c^2k^2 - a^2}}\frac{\sqrt{c^2k^2 - a^2}}{(s+a)^2 + (c^2k^2 - a^2)}$$

Now we're ready to take the inverse Laplace transform. Use equation (6) here.

$$U(k,t) = \frac{G(k)}{\sqrt{c^2k^2 - a^2}} e^{-at} \sin\sqrt{c^2k^2 - a^2}t$$

To make $U(k,t)$ easier to work with, introduce a new variable $\omega = \omega(k)$ for the square root term.

$$\omega(k) = \sqrt{c^2k^2 - a^2}$$

It's not necessary to consider the case where $c^2k^2 - a^2 < 0$ because $-i \sin i\omega t = \sinh \omega t$. We're ready now to take the inverse Fourier transform. It is defined as

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) e^{ikx}\, dk.$$

Plug $U(k,t)$ into the definition of the inverse Fourier transform to get $u(x,t)$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{\omega} e^{-at} \sin\omega t\, e^{ikx}\, dk$$

Recall that sine can be written in terms of exponentials using Euler's formula.

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

Substituting this expression and bringing e^{-at} in front of the integral, we get

$$u(x,t) = \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{\omega} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} e^{ikx}\, dk.$$

Distribute the terms in the integrand.

$$u(x,t) = \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{2i\omega} e^{i(kx+\omega t)} - \frac{G(k)}{2i\omega} e^{i(kx-\omega t)}\, dk.$$

Therefore,

$$u(x,t) = \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) e^{i(kx+\omega t)} + B(k) e^{i(kx-\omega t)} \right] dk,$$

where

$$\begin{aligned} \omega = \omega(k) &= \sqrt{c^2k^2 - a^2} \\ A(k) &= \frac{G(k)}{2i\omega} \\ B(k) &= -\frac{G(k)}{2i\omega} \\ G(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx. \end{aligned}$$

Comparing this with the solution to Exercise 1.12, we see that we could've gotten the same result by replacing α with $2a$ and $F(k)$ with 0, as expected.