Exercise 41

Use the joint Laplace and Fourier transform to solve the initial-value problem for transient water waves which satisfies (see Debnath 1994, p. 92)

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty,\ -\infty < z < 0,\ t > 0,$$

$$\left.\begin{aligned} &\phi_z = \eta_t, \\ &\phi_t + g\eta = -\frac{P}{\rho}p(x)e^{i\omega t} \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0,$$

$$\phi(x, z, 0) = 0 = \eta(x, 0) \quad \text{for all } x \text{ and } z,$$

where P and ρ are constants.

Solution

The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x, z, t)\} = \Phi(k, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\phi(x, z, t)\, dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial x^n}\right\} = (ik)^n\Phi(k, z, t)$$

$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial z^n}\right\} = \frac{d^n\Phi}{dz^n}$$

$$\mathcal{F}_x\left\{\frac{\partial^n\phi}{\partial t^n}\right\} = \frac{d^n\Phi}{dt^n}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx} + \phi_{zz}\} = \mathcal{F}_x\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2\Phi + \frac{d^2\Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2\Phi}{dz^2} = k^2\Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$$

In order for Φ to remain bounded as $z \to -\infty$, we require that $B(k,t) = 0$. So we have

$$\Phi(k,z,t) = A(k,t)e^{|k|z}. \tag{1}$$

Take the Fourier transform with respect to x of the boundary conditions now.

$$\begin{aligned} \mathcal{F}_x\{\phi_z\} &= \mathcal{F}_x\{\eta_t\} \\ \mathcal{F}_x\{\phi_t + g\eta\} &= \mathcal{F}_x\left\{-\frac{P}{\rho}p(x)e^{i\omega t}\right\} \end{aligned}$$

Use the linearity property.

$$\begin{aligned} \mathcal{F}_x\{\phi_z\} &= \mathcal{F}_x\{\eta_t\} \\ \mathcal{F}_x\{\phi_t\} + g\mathcal{F}_x\{\eta\} &= -\frac{P}{\rho}e^{i\omega t}\mathcal{F}_x\{p(x)\} \end{aligned}$$

Transform the partial derivatives.

$$\begin{aligned} \frac{d\Phi}{dz} &= \frac{dH}{dt} \\ \frac{d\Phi}{dt} + gH &= -\frac{P}{\rho}e^{i\omega t}\tilde{p}(k) \end{aligned}$$

Plug in the expression for Φ in equation (1) into these equations. These two equations hold at the boundary, so we have to evaluate these terms at $z = 0$.

$$A(k,t)|k| = \frac{dH}{dt} \tag{2}$$

$$\frac{\partial A}{\partial t} + gH = -\frac{P}{\rho}e^{i\omega t}\tilde{p}(k) \tag{3}$$

We now have a system of two equations for two unknowns, A and H. Differentiate both sides of equation (3) with respect to t.

$$\begin{aligned} A(k,t)|k| &= \frac{dH}{dt} \\ \frac{\partial^2 A}{\partial t^2} + g\frac{dH}{dt} &= -\frac{i\omega P}{\rho}e^{i\omega t}\tilde{p}(k) \end{aligned}$$

Substitute the first equation into the second.

$$\frac{\partial^2 A}{\partial t^2} + g|k|A = -\frac{i\omega P}{\rho}e^{i\omega t}\tilde{p}(k) \tag{4}$$

This is a second-order inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$A = A_c + A_p$$

A_c is the solution to the associated homogeneous equation,

$$\frac{\partial^2 A}{\partial t^2} + g|k|A = 0,$$

which has the solution

$$A_c = C_1(k)\cos\sqrt{g|k|}t + C_2(k)\sin\sqrt{g|k|}t.$$

The inhomogeneous term is an exponential, so A_p has the form $C_3(k)e^{i\omega t}$. Plug this form into equation (4) to determine $C_3(k)$.

$$-C_3(k)(\omega^2 - g|k|)e^{i\omega t} = -\frac{i\omega P}{\rho}e^{i\omega t}\tilde{p}(k) \quad \rightarrow \quad C_3(k) = \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}$$

The general solution to equation (4) is thus

$$A(k,t) = C_1(k)\cos\sqrt{g|k|}t + C_2(k)\sin\sqrt{g|k|}t + \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}e^{i\omega t}.$$

We use the provided initial conditions, $\phi(x,z,0) = 0$ and $\eta(x,0) = 0$, to determine $C_1(k)$ and $C_2(k)$. Take the Fourier transform of both sides of them.

$$\begin{aligned} \phi(x,z,0) = 0 \quad \rightarrow \quad \mathcal{F}_x\{\phi(x,z,0)\} &= \mathcal{F}_x\{0\} \\ \Phi(k,z,0) &= 0 \qquad (5) \\ \eta(x,0) = 0 \quad \rightarrow \quad \mathcal{F}_x\{\eta(x,0)\} &= \mathcal{F}_x\{0\} \\ H(k,0) &= 0 \qquad (6) \end{aligned}$$

Substituting $t = 0$ into equation (1) and using equation (5), we obtain

$$\Phi(k,z,0) = A(k,0)e^{|k|z} = 0 \quad \rightarrow \quad A(k,0) = 0.$$

We can now determine $C_1(k)$.

$$A(k,0) = C_1(k) + \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)} = 0 \quad \rightarrow \quad C_1(k) = -\frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}$$

Solve equation (3) for $H(k,t)$.

$$H(k,t) = -\frac{1}{g}\left[\frac{\partial A}{\partial t} + \frac{P}{\rho}\tilde{p}(k)e^{i\omega t}\right]$$

Using equation (6) and solving the resulting equation for $C_2(k)$ yields

$$C_2(k) = \frac{\sqrt{g|k|}P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}.$$

With $C_1(k)$ and $C_2(k)$ determined, $A(k,t)$ is known and consequently $H(k,t)$ and $\Phi(k,z,t)$ are as well.

$$\Phi(k,z,t) = \left[-\frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}\cos\sqrt{g|k|}t + \frac{\sqrt{g|k|}P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}\sin\sqrt{g|k|}t + \frac{i\omega P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}e^{i\omega t}\right]e^{|k|z}$$

$$H(k,t) = \frac{1}{\rho(g|k| - \omega^2)}\left[P\tilde{p}(k)|k|\left(-e^{i\omega t} + \cos\sqrt{g|k|}t\right) + \frac{iP\tilde{p}(k)\omega\sqrt{|k|}}{\sqrt{g}}\sin\sqrt{g|k|}t\right]$$

Factoring $\Phi(k,z,t)$ and $H(k,t)$ gives

$$\Phi(k,z,t) = \frac{P\tilde{p}(k)}{\rho(\omega^2 - g|k|)}\left[i\omega\left(e^{i\omega t} - \cos\sqrt{g|k|}t\right) + \sqrt{g|k|}\sin\sqrt{g|k|}t\right]e^{|k|z}$$
$$H(k,t) = \frac{P\tilde{p}(k)|k|}{\rho(g|k| - \omega^2)}\left[-e^{i\omega t} + \cos\sqrt{g|k|}t + \frac{i\omega}{\sqrt{g|k|}}\sin\sqrt{g|k|}t\right].$$

Taking the inverse Fourier transform of $\Phi(k,z,t)$ and $H(k,t)$ gives us $\phi(x,z,t)$ and $\eta(x,t)$, respectively.

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k,z,t)e^{ikx}\,dk$$
$$\eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(k,t)e^{ikx}\,dk$$

Note that $\tilde{p}(k)$ is the Fourier transform of $p(x)$.

$$\tilde{p}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}p(x)\,dx$$