Exercise 43

(a) The axisymmetric initial-value problem is governed by

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + \delta(t)f(r), \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = 0 \quad \text{for } 0 < r < \infty.$$

Show that the formal solution of this problem is

$$u(r,t) = \int_0^\infty kJ_0(kr)\tilde{f}(k)\exp(-k^2\kappa t)\,dk.$$

(b) When $f(r) = \frac{Q}{\pi a^2}H(a-r)$, show that the solution is

$$u(r,t) = \frac{Q}{\pi a}\int_0^\infty J_0(kr)J_1(ak)\exp(-k^2\kappa t)\,dk.$$

Solution

Part (a)

The PDE is defined for $0 < r < \infty$, so the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r,z)\} = \tilde{u}(k,z) = \int_0^\infty rJ_0(kr)u(r,z)\,dr,$$

where $J_0(kr)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} = -k^2\tilde{u}(k,z)$$

The partial derivative with respect to t transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n\tilde{u}}{dt^n}$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\{u_t\} = \mathcal{H}_0\left\{\kappa\left(u_{rr} + \frac{1}{r}u_r\right) + \delta(t)f(r)\right\}$$

The Hankel transform is a linear operator.

$$\mathcal{H}_0\{u_t\} = \kappa\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r\right\} + \delta(t)\mathcal{H}_0\{f(r)\}$$

Use the relations above to transform the derivatives.

$$\frac{d\tilde{u}}{dt} = -\kappa k^2\tilde{u} + \delta(t)\tilde{f}(k) \tag{1}$$

The PDE has thus been reduced to an ODE. For $t > 0$, $\delta(t) = 0$ and the ODE becomes

$$\frac{d\tilde{u}}{dt} = -\kappa k^2 \tilde{u},$$

which has the solution

$$\tilde{u}(k,t) = A(k)e^{-\kappa k^2 t}.$$

To determine $A(k)$, we have to use the provided initial condition. Take the zero-order Hankel transform of both sides of it.

$$\begin{aligned} u(r,0) = 0 \quad \rightarrow \quad \mathcal{H}_0\{u(r,0)\} &= \mathcal{H}_0\{0\} \\ \tilde{u}(k,0) &= 0 \end{aligned} \tag{2}$$

Because of the delta function in equation (1), equation (2) is not what we will use. Integrate both sides of equation (1) with respect to t from $t = -\varepsilon$ to $t = \varepsilon$.

$$\int_{-\varepsilon}^{\varepsilon} \frac{d\tilde{u}}{dt}\,dt = -\int_{-\varepsilon}^{\varepsilon} \kappa k^2 \tilde{u}\,dt + \int_{-\varepsilon}^{\varepsilon} \delta(t)\tilde{f}(k)\,dt$$

Bring the constants out in front of the second and third integrals and evaluate the first one.

$$\tilde{u}(k,\varepsilon) - \tilde{u}(k,-\varepsilon) = -\kappa k^2 \int_{-\varepsilon}^{\varepsilon} \tilde{u}\,dt + \tilde{f}(k) \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt$$

The integral of $\tilde{u}$ over an infinitesimally small interval is 0, and the integral of the delta function is 1.

$$\tilde{u}(k,\varepsilon) - \tilde{u}(k,-\varepsilon) = \tilde{f}(k) \tag{3}$$

Because of equation (2), $\tilde{u}(k,-\varepsilon) = 0$. As a result of the delta function in the ODE, $\tilde{u}$ jumps from 0 at $t = 0$ to $\tilde{f}(k)$ the instant after and falls off exponentially. Hence,

$$\tilde{u}(k,t) = \tilde{f}(k)e^{-\kappa k^2 t}H(t).$$

Since we're only interested in the solution for $t > 0$, we can drop the Heaviside function.

$$\tilde{u}(k,t) = \tilde{f}(k)e^{-\kappa k^2 t}, \quad t > 0$$

We can get $u(r,t)$ by taking the inverse Hankel transform of this.

$$u(r,t) = \mathcal{H}_0^{-1}\{\tilde{u}(k,t)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(k,t)\} = \int_0^\infty kJ_0(kr)\tilde{u}(k,t)\,dk.$$

Therefore,

$$u(r,t) = \int_0^\infty kJ_0(kr)\tilde{f}(k)e^{-\kappa k^2 t}\,dk.$$

Part (b)

If

$$f(r) = \frac{Q}{\pi a^2} H(a - r),$$

then

$$\tilde{f}(k) = \mathcal{H}_0 \left\{ \frac{Q}{\pi a^2} H(a - r) \right\}.$$

The Hankel transform is a linear operator.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \mathcal{H}_0 \{H(a - r)\}$$

Use the definition of the zero-order Hankel transform.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_0^\infty r J_0(kr) H(a - r)\, dr$$

Make the substitution $w = a - r$.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_a^{-\infty} (a - w) J_0[k(a - w)] H(w)\, (-dw)$$

Use the minus sign to switch the limits of integration.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_{-\infty}^a (a - w) J_0[k(a - w)] H(w)\, dw$$

The Heaviside function is equal to 1 when $w > 0$ and is equal to 0 when $w < 0$.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_0^a (a - w) J_0[k(a - w)]\, dw$$

Make the substitution $p = a - w$.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_a^0 p J_0(kp)\, (-dp)$$

Use the minus sign to switch the limits of integration.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \int_0^a p J_0(kp)\, dp$$

Look up this integral in a table.

$$\tilde{f}(k) = \frac{Q}{\pi a^2} \cdot \frac{a}{k} J_1(ka)$$

Simplify the result.

$$\tilde{f}(k) = \frac{Q}{\pi a k} J_1(ka).$$

Substituting this expression for $\tilde{f}(k)$ in the solution for $u(r, t)$, we get

$$u(r, t) = \int_0^\infty k J_0(kr) \frac{Q}{\pi a k} J_1(ka) e^{-\kappa k^2 t}\, dk.$$

Therefore,

$$u(r, t) = \frac{Q}{\pi a} \int_0^\infty e^{-\kappa k^2 t} J_0(kr) J_1(ka)\, dk.$$