Exercise 47

Solve the nonhomogeneous diffusion problem

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = f(r), \quad 0 < r < \infty,$$

where κ is a constant.

Solution

Since $0 < r < \infty$, the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r,t)\} = \tilde{u}(k,t) = \int_0^\infty r J_0(kr)u(r,t)\,dr,$$

where $J_0(\kappa r)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} = -k^2\tilde{u}(k,z)$$

The partial derivative with respect to t transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n\tilde{u}}{dt^n}$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\{u_t\} = \mathcal{H}_0\left\{\kappa\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t)\right\}$$

The Hankel transform is a linear operator.

$$\mathcal{H}_0\{u_t\} = \kappa\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r\right\} + \mathcal{H}_0\{Q(r,t)\}$$

Use the relations above to transform the partial derivatives.

$$\frac{d\tilde{u}}{dt} = -\kappa k^2\tilde{u} + \tilde{Q}(k,t)$$

Move the term with $\tilde{u}$ to the other side.

$$\frac{d\tilde{u}}{dt} + \kappa k^2\tilde{u} = \tilde{Q}(k,t)$$

The PDE has thus been reduced to a first-order inhomogeneous ODE that can be solved with an integrating factor.

$$I = e^{\int^t \kappa k^2\,ds} = e^{\kappa k^2 t}$$

Multiply both sides of the ODE by I.

$$e^{\kappa k^2 t}\frac{d\tilde{u}}{dt} + \kappa k^2 e^{\kappa k^2 t}\tilde{u} = e^{\kappa k^2 t}\tilde{Q}(k,t)$$

The left side can be written as $d/dt(I\tilde{u})$ as a result of the product rule.

$$\frac{d}{dt}\left(e^{\kappa k^2 t}\tilde{u}\right) = e^{\kappa k^2 t}\tilde{Q}(k,t)$$

Integrate both sides with respect to t.

$$e^{\kappa k^2 t}\tilde{u} = \int_0^t e^{\kappa k^2 s}\tilde{Q}(k,s)\,ds + C(k) \tag{1}$$

The lower limit of integration is arbitrary. $C(k)$ will be adjusted to match the initial condition, $u(r,0) = f(r)$. Take the zero-order Hankel transform of both sides of it.

$$\begin{aligned}\mathcal{H}_0\{u(r,0)\} &= \mathcal{H}_0\{f(r)\} \\ \tilde{u}(k,0) &= \tilde{f}(k)\end{aligned} \tag{2}$$

Plug in $t = 0$ into equation (1) and use equation (2) to determine $C(k)$.

$$\tilde{u}(k,0) = C(k) = \tilde{f}(k)$$

Dividing both sides of equation (1) by $e^{\kappa k^2 t}$, we therefore have

$$\tilde{u}(k,t) = e^{-\kappa k^2 t}\left[\int_0^t e^{\kappa k^2 s}\tilde{Q}(k,s)\,ds + \tilde{f}(k)\right].$$

Now that we have $\tilde{u}(k,t)$, we can change back to $u(r,t)$ by taking the inverse Hankel transform of it.

$$u(r,t) = \mathcal{H}_0^{-1}\{\tilde{u}(k,t)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(k,t)\} = \int_0^\infty kJ_0(kr)\tilde{u}(k,t)\,dk.$$

Therefore,

$$u(r,t) = \int_0^\infty kJ_0(kr)e^{-\kappa k^2 t}\left[\int_0^t e^{\kappa k^2 s}\tilde{Q}(k,s)\,ds + \tilde{f}(k)\right]dk,$$

where

$$\begin{aligned}\tilde{f}(k) &= \int_0^\infty rJ_0(kr)f(r)\,dr \\ \tilde{Q}(k,t) &= \int_0^\infty Q(r,t)J_0(kr)r\,dr.\end{aligned}$$