# Problem 1.1

Using the definitions in Eqs. 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive:

(a) when the three vectors are coplanar;

(b) in the general case.

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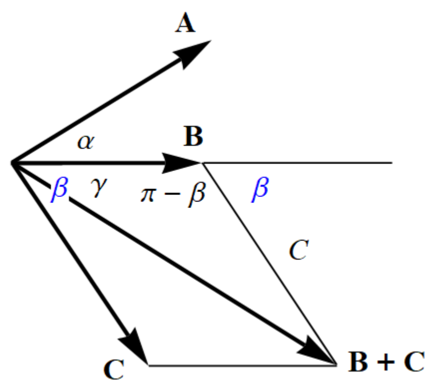
**Solution**

## Part (a)

Consider three vectors, **A** and **B** and **C**, in the same plane. Let $\alpha$ be the angle between **A** and **B**, let $\beta$ be the angle between **B** and **C**, and let $\gamma$ be the angle between **B** and **B** + **C**.




The aim here is to show that the dot product is distributive.

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) \stackrel{?}{=} \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

Use the definition in Eq. 1.1 to rewrite both sides.

$$A|\mathbf{B} + \mathbf{C}| \cos \theta_{ABC} \stackrel{?}{=} AB \cos \theta_{AB} + AC \cos \theta_{AC}$$

$$A|\mathbf{B} + \mathbf{C}| \cos(\alpha + \gamma) \stackrel{?}{=} AB \cos \alpha + AC \cos(\alpha + \beta)$$

$$A|\mathbf{B} + \mathbf{C}|(\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \stackrel{?}{=} AB \cos \alpha + AC(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \tag{1}$$

Use the law of cosines to find a formula for $|\mathbf{B} + \mathbf{C}|$.

$$|\mathbf{B} + \mathbf{C}|^2 = B^2 + C^2 - 2BC \cos(\pi - \beta)$$

$$|\mathbf{B} + \mathbf{C}|^2 = B^2 + C^2 + 2BC \cos \beta$$

$$|\mathbf{B} + \mathbf{C}| = \sqrt{B^2 + C^2 + 2BC \cos \beta} \tag{2}$$

Use the law of cosines again to determine $\gamma$.

$$C^2 = B^2 + |\mathbf{B}+\mathbf{C}|^2 - 2B|\mathbf{B}+\mathbf{C}|\cos\gamma$$

$$\cos\gamma = \frac{B^2 + |\mathbf{B}+\mathbf{C}|^2 - C^2}{2B|\mathbf{B}+\mathbf{C}|}$$

$$\cos\gamma = \frac{B^2 + (B^2 + C^2 + 2BC\cos\beta) - C^2}{2B\sqrt{B^2 + C^2 + 2BC\cos\beta}}$$

$$\cos\gamma = \frac{2B^2 + 2BC\cos\beta}{2B\sqrt{B^2 + C^2 + 2BC\cos\beta}}$$

$$\cos\gamma = \frac{B + C\cos\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}} \tag{3}$$

$$\cos^2\gamma = \frac{B^2 + C^2\cos^2\beta + 2BC\cos\beta}{B^2 + C^2 + 2BC\cos\beta}$$

$$1 - \sin^2\gamma = \frac{B^2 + C^2\cos^2\beta + 2BC\cos\beta}{B^2 + C^2 + 2BC\cos\beta}$$

$$\sin^2\gamma = 1 - \frac{B^2 + C^2\cos^2\beta + 2BC\cos\beta}{B^2 + C^2 + 2BC\cos\beta}$$

$$\sin^2\gamma = \frac{(B^2 + C^2 + 2BC\cos\beta) - (B^2 + C^2\cos^2\beta + 2BC\cos\beta)}{B^2 + C^2 + 2BC\cos\beta}$$

$$\sin^2\gamma = \frac{C^2(1 - \cos^2\beta)}{B^2 + C^2 + 2BC\cos\beta}$$

$$\sin^2\gamma = \frac{C^2\sin^2\beta}{B^2 + C^2 + 2BC\cos\beta}$$

$$\sin\gamma = \frac{C\sin\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}} \tag{4}$$

Now plug equations (2), (3), and (4) into the left side of equation (1) and see if the right side is obtained.

$A|\mathbf{B}+\mathbf{C}|(\cos\alpha\cos\gamma - \sin\alpha\sin\gamma)$

$$= A\sqrt{B^2 + C^2 + 2BC\cos\beta}\left(\cos\alpha\frac{B + C\cos\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}} - \sin\alpha\frac{C\sin\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}}\right)$$

$$= A\left[\cos\alpha(B + C\cos\beta) - \sin\alpha(C\sin\beta)\right]$$

$$= AB\cos\alpha + AC(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

Therefore, the dot product is distributive when the three vectors are coplanar.

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

The aim now is to show that the cross product is distributive.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \stackrel{?}{=} \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

Use the definition in Eq. 1.4 to rewrite both sides.

$$A|\mathbf{B} + \mathbf{C}| \sin\theta_{ABC}\,\hat{\mathbf{n}} \stackrel{?}{=} AB\sin\theta_{AB}\,\hat{\mathbf{n}} + AC\sin\theta_{AC}\,\hat{\mathbf{n}}$$

$$A|\mathbf{B} + \mathbf{C}| \sin(\alpha+\gamma)\,\hat{\mathbf{n}} \stackrel{?}{=} AB\sin\alpha\,\hat{\mathbf{n}} + AC\sin(\alpha+\beta)\,\hat{\mathbf{n}}$$

$$A|\mathbf{B} + \mathbf{C}|(\sin\alpha\cos\gamma + \cos\alpha\sin\gamma) \stackrel{?}{=} AB\sin\alpha + AC(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \tag{5}$$

Plug equations (2), (3), and (4) into the left side of equation (5) and see if the right side is obtained.

$$\begin{aligned}
&A|\mathbf{B} + \mathbf{C}|(\sin\alpha\cos\gamma + \cos\alpha\sin\gamma) \\
&\quad= A\sqrt{B^2 + C^2 + 2BC\cos\beta}\left(\sin\alpha\frac{B + C\cos\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}} + \cos\alpha\frac{C\sin\beta}{\sqrt{B^2 + C^2 + 2BC\cos\beta}}\right) \\
&\quad= A\left[\sin\alpha(B + C\cos\beta) + \cos\alpha(C\sin\beta)\right] \\
&\quad= AB\sin\alpha + AC(\sin\alpha\cos\beta + \cos\alpha\sin\beta)
\end{aligned}$$

Therefore, the cross product is distributive when the three vectors are coplanar.

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

## Part (b)

Prove that the dot product is distributive generally.

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j B_j\right) + \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j C_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left[\sum_{j=1}^{3} (\boldsymbol{\delta}_j B_j + \boldsymbol{\delta}_j C_j)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left[\sum_{j=1}^{3} \boldsymbol{\delta}_j (B_j + C_j)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_i (B_j + C_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j)(A_i B_j + A_i C_j)
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)A_iB_j + \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)A_iC_j \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right) + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jC_j\right) \\
&= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}
\end{aligned}$$

Prove that the cross product is distributive generally.

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right)+\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jC_j\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\times\left[\sum_{j=1}^{3}(\boldsymbol{\delta}_jB_j+\boldsymbol{\delta}_jC_j)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\times\left[\sum_{j=1}^{3}\boldsymbol{\delta}_j(B_j+C_j)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)A_i(B_j+C_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)(A_iB_j+A_iC_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)A_iB_j + \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)A_iC_j \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right) + \left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jC_j\right) \\
&= \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}
\end{aligned}$$