Problem 1.13

Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$;

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

Solution

Write a formula for the separation vector.

$$\boldsymbol{\mathscr{r}} = (x, y, z) - (x', y', z') = (x - x', y - y', z - z') = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

Its magnitude is then

$$|\boldsymbol{\mathscr{r}}| = \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

Calculate the first gradient.

$$\begin{aligned}
\nabla(\mathscr{r}^2) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)[(x - x')^2 + (y - y')^2 + (z - z')^2] \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}[(x - x')^2 + (y - y')^2 + (z - z')^2] \\
&\qquad + \hat{\mathbf{y}}\frac{\partial}{\partial y}[(x - x')^2 + (y - y')^2 + (z - z')^2] \\
&\qquad\qquad + \hat{\mathbf{z}}\frac{\partial}{\partial z}[(x - x')^2 + (y - y')^2 + (z - z')^2] \\
&= \hat{\mathbf{x}}[2(x - x')] + \hat{\mathbf{y}}[2(y - y')] + \hat{\mathbf{z}}[2(z - z')] \\
&= 2\left[(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}\right] \\
&= 2\boldsymbol{\mathscr{r}}
\end{aligned}$$

Calculate the second gradient.

$$\begin{aligned}
\nabla\left(\frac{1}{\mathscr{r}}\right) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left[\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}\right] \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2} \\
&\qquad + \hat{\mathbf{y}}\frac{\partial}{\partial y}\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2} \\
&\qquad\qquad + \hat{\mathbf{z}}\frac{\partial}{\partial z}\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2} \\
&= \hat{\mathbf{x}}\left(-\frac{1}{2}\right)\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-3/2} \cdot 2(x - x') \\
&\qquad + \hat{\mathbf{y}}\left(-\frac{1}{2}\right)\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-3/2} \cdot 2(y - y') \\
&\qquad\qquad + \hat{\mathbf{z}}\left(-\frac{1}{2}\right)\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-3/2} \cdot 2(z - z')
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla\left(\frac{1}{\mathscr{r}}\right) &= \hat{\mathbf{x}}\left\{-\frac{x-x'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} \\
&\quad + \hat{\mathbf{y}}\left\{-\frac{y-y'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} \\
&\quad\quad + \hat{\mathbf{z}}\left\{-\frac{z-z'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\} \\
&= -\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2}\left[\hat{\mathbf{x}}\frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right. \\
&\quad + \hat{\mathbf{y}}\frac{y-y'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
&\quad \left. + \hat{\mathbf{z}}\frac{z-z'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right] \\
&= -\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2}\left[\frac{(x-x')\hat{\mathbf{x}}+(y-y')\hat{\mathbf{y}}+(z-z')\hat{\mathbf{z}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right] \\
&= -\frac{1}{\mathscr{r}^2}\left(\frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|}\right) \\
&= -\frac{1}{\mathscr{r}^2}(\hat{\boldsymbol{\mathscr{r}}}) \\
&= -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}
\end{aligned}$$

Calculate the third gradient.

$$\begin{aligned}
\nabla(\mathscr{r}^n) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^n \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2} \\
&\quad + \hat{\mathbf{y}}\frac{\partial}{\partial y}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2} \\
&\quad\quad + \hat{\mathbf{z}}\frac{\partial}{\partial z}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2} \\
&= \hat{\mathbf{x}}\left(\frac{n}{2}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2-1}\cdot 2(x-x') \\
&\quad + \hat{\mathbf{y}}\left(\frac{n}{2}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2-1}\cdot 2(y-y') \\
&\quad\quad + \hat{\mathbf{z}}\left(\frac{n}{2}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2-1}\cdot 2(z-z') \\
&= n\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{n-2}{2}}\left[(x-x')\hat{\mathbf{x}}+(y-y')\hat{\mathbf{y}}+(z-z')\hat{\mathbf{z}}\right]
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla(\mathscr{r}^n) &= n\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^{n-2}\left[(x-x')\hat{\mathbf{x}}+(y-y')\hat{\mathbf{y}}+(z-z')\hat{\mathbf{z}}\right] \\
&= n\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^{n-1}\left[\frac{(x-x')\hat{\mathbf{x}}+(y-y')\hat{\mathbf{y}}+(z-z')\hat{\mathbf{z}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right] \\
&= n\mathscr{r}^{n-1}\left(\frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|}\right) \\
&= n\mathscr{r}^{n-1}\hat{\boldsymbol{\mathscr{r}}}
\end{aligned}$$