Problem 1.17

In two dimensions, show that the divergence transforms as a scalar under rotations. [*Hint:* Use Eq. 1.29 to determine $\bar{v}_y$ and $\bar{v}_z$, and the method of Prob. 1.14 to calculate the derivatives. Your aim is to show that $\partial\bar{v}_y/\partial\bar{y} + \partial\bar{v}_z/\partial\bar{z} = \partial v_y/\partial y + \partial v_z/\partial z$.]

Solution

Suppose there's a yz-coordinate system, and the axes are rotated counterclockwise by an angle ϕ in order to make a new $\bar{y}\bar{z}$-coordinate system. The position vector $\mathbf{r} = \langle y, z\rangle$ and vector $\mathbf{v} = \langle v_y, v_z\rangle$ transform under the rotation to $\bar{\mathbf{r}} = \langle \bar{y}, \bar{z}\rangle$ and $\bar{\mathbf{v}} = \langle \bar{v}_y, \bar{v}_z\rangle$.

$$\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix} \qquad \begin{pmatrix}\bar{v}_y\\ \bar{v}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}v_y\\ v_z\end{pmatrix}$$

$$\begin{cases}\bar{y} = y\cos\phi + z\sin\phi\\ \bar{z} = -y\sin\phi + z\cos\phi\end{cases} \qquad \begin{cases}\bar{v}_y = v_y\cos\phi + v_z\sin\phi\\ \bar{v}_z = -v_y\sin\phi + v_z\cos\phi\end{cases}$$

Solve the system on the left for y and z.

$$y = \bar{y}\cos\phi - \bar{z}\sin\phi$$

$$z = \bar{y}\sin\phi + \bar{z}\cos\phi$$

Use the chain rule to determine the derivatives in the new coordinate system.

$$\frac{\partial}{\partial\bar{y}} = \frac{\partial y}{\partial\bar{y}}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\bar{y}}\frac{\partial}{\partial z} = (\cos\phi)\frac{\partial}{\partial y} + (\sin\phi)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\bar{z}} = \frac{\partial y}{\partial\bar{z}}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\bar{z}}\frac{\partial}{\partial z} = (-\sin\phi)\frac{\partial}{\partial y} + (\cos\phi)\frac{\partial}{\partial z}$$

Now calculate the divergence of $\mathbf{v}$ in the new coordinate system.

$$\begin{aligned}
\overline{\nabla}\cdot\bar{\mathbf{v}} &= \frac{\partial\bar{v}_y}{\partial\bar{y}} + \frac{\partial\bar{v}_z}{\partial\bar{z}}\\
&= \frac{\partial}{\partial\bar{y}}(v_y\cos\phi + v_z\sin\phi) + \frac{\partial}{\partial\bar{z}}(-v_y\sin\phi + v_z\cos\phi)\\
&= \left(\cos\phi\frac{\partial}{\partial y} + \sin\phi\frac{\partial}{\partial z}\right)(v_y\cos\phi + v_z\sin\phi) + \left(-\sin\phi\frac{\partial}{\partial y} + \cos\phi\frac{\partial}{\partial z}\right)(-v_y\sin\phi + v_z\cos\phi)\\
&= \left(\frac{\partial v_y}{\partial y}\cos^2\phi + \cancel{\frac{\partial v_z}{\partial y}\sin\phi\cos\phi} + \cancel{\frac{\partial v_y}{\partial z}\sin\phi\cos\phi} + \frac{\partial v_z}{\partial z}\sin^2\phi\right)\\
&\qquad + \left(\frac{\partial v_y}{\partial y}\sin^2\phi - \cancel{\frac{\partial v_z}{\partial y}\sin\phi\cos\phi} - \cancel{\frac{\partial v_y}{\partial z}\sin\phi\cos\phi} + \frac{\partial v_z}{\partial z}\cos^2\phi\right)
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}\bar{\nabla} \cdot \bar{\mathbf{v}} &= \frac{\partial v_y}{\partial y}(\cos^2\phi + \sin^2\phi) + \frac{\partial v_z}{\partial z}(\sin^2\phi + \cos^2\phi) \\ &= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \nabla \cdot \mathbf{v}\end{aligned}$$

Therefore, the divergence transforms under rotations as a scalar.