Problem 1.19

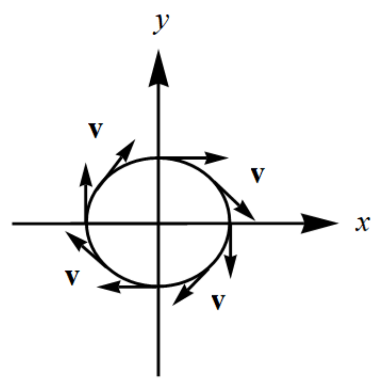
Draw a circle in the xy plane. At a few representative points draw the unit vector $\mathbf{v}$ tangent to the circle, pointing in the clockwise direction. By comparing adjacent vectors, determine the *signs* of $\partial v_x/\partial y$ and $\partial v_y/\partial x$. According to Eq. 1.41 then, what is the direction of $\nabla \times \mathbf{v}$?

Solution

In Cartesian coordinates the curl of $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ is

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \end{aligned} \tag{1.41}$$

The curl is only nonzero if the components of $\mathbf{v}$ change in space. For instance, consider the circle shown below in the xy-plane and a vector $\mathbf{v}$ acting on it tangentially in the clockwise direction.



Since $\mathbf{v}$ lies in and changes in the xy-plane only, the z-derivatives are zero and $v_z = 0$.

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}(0) + \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Notice that the x-component of $\mathbf{v}$ is negative in the bottom half of the circle and becomes positive in the top half. $\partial v_x/\partial y$, which represents the change in v_x as y increases, is positive then. Also, the y-component of $\mathbf{v}$ is positive in the left half and becomes negative in the right half. $\partial v_y/\partial x$, which represents the change in v_y as x increases, is negative then. Therefore, the curl of $\mathbf{v}$ points in the negative z-direction (into the paper).

If the tangent vectors pointed in the counterclockwise direction instead, then the curl of $\mathbf{v}$ would point in the positive z-direction (out of the paper). These results could have been predicted with the right-hand corkscrew rule: curl the four fingers in the direction of $\mathbf{v}$, and the thumb points in the direction of $\nabla \times \mathbf{v}$.