Problem 1.21

Prove product rules (i), (iv), and (v).

Solution

Proof of (i)

The aim is prove that

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{i}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla(fg) &= \sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}(fg) \\
&= \sum_{i=1}^{3} \boldsymbol{\delta}_i \left(\frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i}\right) \\
&= \sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial f}{\partial x_i} g + \sum_{i=1}^{3} \boldsymbol{\delta}_i f\frac{\partial g}{\partial x_i} \\
&= g\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial f}{\partial x_i}\right) + f\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial g}{\partial x_i}\right) \\
&= g(\nabla f) + f(\nabla g) \\
&= f(\nabla g) + g(\nabla f)
\end{aligned}$$

Proof of (iv)

The aim is prove that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{iv}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) A_j B_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} A_j B_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \left(\frac{\partial A_j}{\partial x_i} B_k + A_j \frac{\partial B_k}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{ikj}) A_j \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{lk} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{lj} \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_k) \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_j) \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\
&= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l\right) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i}\right) - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l\right) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \varepsilon_{ikj} \frac{\partial B_k}{\partial x_i}\right)
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l\right) \cdot \left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\frac{\partial A_j}{\partial x_i}\right] - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l\right) \cdot \left[\sum_{i=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_k)\frac{\partial B_k}{\partial x_i}\right] \\
&= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l\right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l\right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right)\right] \\
&= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})
\end{aligned}$$

Proof of (v)

The aim is to prove that

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f). \tag{v}$$

Write out the left side explicitly.

$$\begin{aligned}
\nabla \times (f\mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[f\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j f\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\frac{\partial}{\partial x_i} A_j f \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\left(\frac{\partial A_j}{\partial x_i} f + A_j \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\frac{\partial A_j}{\partial x_i} f + \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) A_j \frac{\partial f}{\partial x_i} \\
&= f\left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j)\frac{\partial A_j}{\partial x_i}\right] + \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) A_j \frac{\partial f}{\partial x_i} \\
&= f\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] + \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial f}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \\
&= f(\nabla \times \mathbf{A}) + (\nabla f) \times \mathbf{A} \\
&= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)
\end{aligned}$$