Problem 1.22

(a) If **A** and **B** are two vector functions, what does the expression $(\mathbf{A}\cdot\nabla)\mathbf{B}$ mean? (That is, what are its x, y, and z components, in terms of the Cartesian components of **A**, **B**, and ∇?)

(b) Compute $(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector defined in Eq. 1.21.

(c) For the functions in Prob. 1.15, evaluate $(\mathbf{v}_a\cdot\nabla)\mathbf{v}_b$.

Solution

Part (a)

Evaluate $(\mathbf{A}\cdot\nabla)\mathbf{B}$ explicitly.

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{B} &= \left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\right]\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right) \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)A_i\frac{\partial}{\partial x_j}\right]\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right) \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}A_i\frac{\partial}{\partial x_j}\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right) \\
&= \left(\sum_{i=1}^{3}A_i\frac{\partial}{\partial x_i}\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k B_k\right) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k A_i\frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\left(\boldsymbol{\delta}_1 A_i\frac{\partial B_1}{\partial x_i}+\boldsymbol{\delta}_2 A_i\frac{\partial B_2}{\partial x_i}+\boldsymbol{\delta}_3 A_i\frac{\partial B_3}{\partial x_i}\right) \\
&= \boldsymbol{\delta}_1 A_1\frac{\partial B_1}{\partial x_1}+\boldsymbol{\delta}_2 A_1\frac{\partial B_2}{\partial x_1}+\boldsymbol{\delta}_3 A_1\frac{\partial B_3}{\partial x_1} \\
&\quad+\boldsymbol{\delta}_1 A_2\frac{\partial B_1}{\partial x_2}+\boldsymbol{\delta}_2 A_2\frac{\partial B_2}{\partial x_2}+\boldsymbol{\delta}_3 A_2\frac{\partial B_3}{\partial x_2} \\
&\quad+\boldsymbol{\delta}_1 A_3\frac{\partial B_1}{\partial x_3}+\boldsymbol{\delta}_2 A_3\frac{\partial B_2}{\partial x_3}+\boldsymbol{\delta}_3 A_3\frac{\partial B_3}{\partial x_3} \\
&= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right) \\
&\qquad+\hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right) \\
&\qquad\qquad+\hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

Part (b)

The radial unit vector from the origin is defined as

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{z}}.$$

Use the formula from part (a) to evaluate $(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}}$.

$$\begin{aligned}
(\hat{\mathbf{r}}\cdot\nabla)\hat{\mathbf{r}} &= \hat{\mathbf{x}}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right] \\
&\quad + \hat{\mathbf{y}}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right] \\
&\quad + \hat{\mathbf{z}}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right] \\
&= \hat{\mathbf{x}}\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\left[\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{x^2+y^2+z^2}}\left[-\frac{xy}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{z}{\sqrt{x^2+y^2+z^2}}\left[-\frac{xz}{(x^2+y^2+z^2)^{3/2}}\right]\right\} \\
&\quad + \hat{\mathbf{y}}\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\left[-\frac{xy}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{x^2+y^2+z^2}}\left[\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{z}{\sqrt{x^2+y^2+z^2}}\left[-\frac{yz}{(x^2+y^2+z^2)^{3/2}}\right]\right\} \\
&\quad + \hat{\mathbf{z}}\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\left[-\frac{xz}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{y}{\sqrt{x^2+y^2+z^2}}\left[-\frac{yz}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{z}{\sqrt{x^2+y^2+z^2}}\left[\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\right]\right\} \\
&= \hat{\mathbf{x}}\left[\frac{xy^2+xz^2}{(x^2+y^2+z^2)^2} - \frac{xy^2}{(x^2+y^2+z^2)^2} - \frac{xz^2}{(x^2+y^2+z^2)^2}\right] \\
&\quad + \hat{\mathbf{y}}\left[-\frac{yx^2}{(x^2+y^2+z^2)^2} + \frac{yx^2+yz^2}{(x^2+y^2+z^2)^2} - \frac{yz^2}{(x^2+y^2+z^2)^2}\right] \\
&\quad + \hat{\mathbf{z}}\left[-\frac{zx^2}{(x^2+y^2+z^2)^2} - \frac{zy^2}{(x^2+y^2+z^2)^2} + \frac{zx^2+zy^2}{(x^2+y^2+z^2)^2}\right] = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = \mathbf{0}
\end{aligned}$$

Part (c)

The three vector functions in Prob. 1.15 are

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

Use the formula from part (a) to evaluate $(\mathbf{v}_a \cdot \nabla)\mathbf{v}_b$.

$$\begin{aligned}
(\mathbf{v}_a \cdot \nabla)\mathbf{v}_b &= \hat{\mathbf{x}}\left[x^2\frac{\partial}{\partial x}(xy) + 3xz^2\frac{\partial}{\partial y}(xy) - 2xz\frac{\partial}{\partial z}(xy)\right] \\
&\qquad + \hat{\mathbf{y}}\left[x^2\frac{\partial}{\partial x}(2yz) + 3xz^2\frac{\partial}{\partial y}(2yz) - 2xz\frac{\partial}{\partial z}(2yz)\right] \\
&\qquad\qquad + \hat{\mathbf{z}}\left[x^2\frac{\partial}{\partial x}(3zx) + 3xz^2\frac{\partial}{\partial y}(3zx) - 2xz\frac{\partial}{\partial z}(3zx)\right] \\
&= \hat{\mathbf{x}}\left[x^2(y) + 3xz^2(x) - 2xz(0)\right] \\
&\qquad + \hat{\mathbf{y}}\left[x^2(0) + 3xz^2(2z) - 2xz(2y)\right] \\
&\qquad\qquad + \hat{\mathbf{z}}\left[x^2(3z) + 3xz^2(0) - 2xz(3x)\right] \\
&= (x^2y + 3x^2z^2)\hat{\mathbf{x}} + (6xz^3 - 4xyz)\hat{\mathbf{y}} - 3x^2z\hat{\mathbf{z}}
\end{aligned}$$