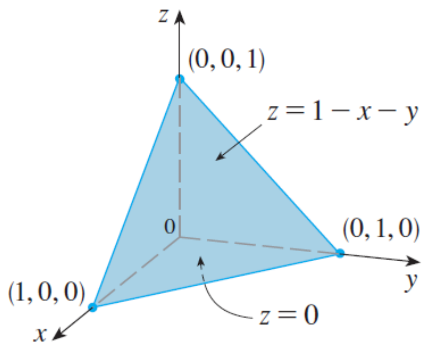
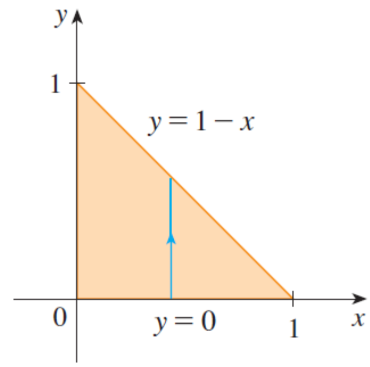
Problem 1.31

Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Solution

Begin by drawing the tetrahedron formed by these four points. Note that the equation of the plane going through $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is $x + y + z = 1$.





The bounding surfaces of the tetrahedron in z are $z = 0$ and $z = 1 - x - y$, the bounding curves of this tetrahedron's projection onto the xy-plane in y are $y = 0$ and $y = 1 - x$, and the bounding limits of this projection's projection onto the x-axis are $x = 0$ and $x = 1$. Therefore,

$$
\begin{aligned}
\iiint\limits_{\text{tetrahedron}} z^2\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z^2\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \left.\frac{z^3}{3}\right|_0^{1-x-y} dy\,dx \\
&= \frac{1}{3}\int_0^1 \int_0^{1-x} (1-x-y)^3\,dy\,dx \\
&= \frac{1}{3}\int_0^1 \int_{1-x}^{0} (u)^3\,(-du)\,dx \\
&= \frac{1}{3}\int_0^1 \int_0^{1-x} u^3\,du\,dx \\
&= \frac{1}{3}\int_0^1 \left.\frac{u^4}{4}\right|_0^{1-x} dx \\
&= \frac{1}{12}\int_0^1 (1-x)^4\,dx \\
&= \frac{1}{12}\int_1^0 (v)^4\,(-dv) = \frac{1}{12}\int_0^1 v^4\,dv = \frac{1}{12}\left(\frac{1}{5}\right) = \frac{1}{60}.
\end{aligned}
$$