Problem 1.34

Test Stokes' theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$, using the triangular shaded area of Fig. 1.34.

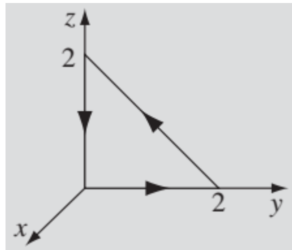


Fig. 1.34

Solution

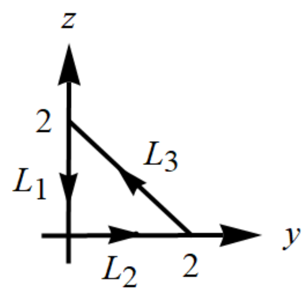
The aim here is to verify Stokes's theorem, which states that

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l},$$

for the given triangular area S. Start by evaluating the surface integral on the left, noting that the area element points in the positive x-direction by the right-hand corkscrew rule.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \int_0^2 \int_0^{2-y} (\nabla \times \mathbf{v}) \cdot (\hat{\mathbf{x}}\, dz\, dy) \\
&= \int_0^2 \int_0^{2-y} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \cdot (\hat{\mathbf{x}}\, dz\, dy) \\
&= \int_0^2 \int_0^{2-y} \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} dz\, dy \\
&= \int_0^2 \int_0^{2-y} \left[\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz) \right] dz\, dy \\
&= \int_0^2 \int_0^{2-y} [(0) - (2y)]\, dz\, dy \\
&= \int_0^2 (-2y)(2-y)\, dy \\
&= -\frac{8}{3}
\end{aligned}$$

Let L_1 be the line segment along the z-axis, let L_2 be the line segment along the y-axis, and let L_3 be the line segment along the hypotenuse.



The integration paths are parameterized as follows.

$$\begin{aligned} \mathbf{l}_1(t) &= \langle 0, 0, 2-t\rangle, & 0 \le t \le 2 \\ \mathbf{l}_2(t) &= \langle 0, t, 0\rangle, & 0 \le t \le 2 \\ \mathbf{l}_3(t) &= \langle 0, 2-t, t\rangle, & 0 \le t \le 2 \end{aligned}$$

Therefore, since $\mathbf{v} = \langle xy, 2yz, 3zx\rangle$, the closed loop integral over the triangular area's boundary is

$$\begin{aligned} \oint_{\text{bdy } S} \mathbf{v}\cdot d\mathbf{l} &= \int_{L_1} \mathbf{v}\cdot d\mathbf{l} + \int_{L_2} \mathbf{v}\cdot d\mathbf{l} + \int_{L_3} \mathbf{v}\cdot d\mathbf{l} \\ &= \int_0^2 \mathbf{v}(\mathbf{l}_1(t))\cdot \mathbf{l}_1'(t)\,dt + \int_0^2 \mathbf{v}(\mathbf{l}_2(t))\cdot \mathbf{l}_2'(t)\,dt + \int_0^2 \mathbf{v}(\mathbf{l}_3(t))\cdot \mathbf{l}_3'(t)\,dt \\ &= \int_0^2 \langle (0)(0), 2(0)(2-t), 3(2-t)(0)\rangle \cdot \langle 0,0,-1\rangle\,dt \\ &\qquad + \int_0^2 \langle (0)(t), 2(t)(0), 3(0)(0)\rangle \cdot \langle 0,1,0\rangle\,dt \\ &\qquad\qquad + \int_0^2 \langle (0)(2-t), 2(2-t)(t), 3(t)(0)\rangle \cdot \langle 0,-1,1\rangle\,dt \\ &= \int_0^2 (0)\,dt + \int_0^2 (0)\,dt + \int_0^2 (-2t)(2-t)\,dt \\ &= 0 + 0 + \left(-\frac{8}{3}\right) \\ &= -\frac{8}{3}. \end{aligned}$$

Because the left and right sides are the same, Stokes's theorem is verified.