Problem 1.35

Check Corollary 1 by using the same function and boundary line as in Ex. 1.11, but integrating over the five faces of the cube in Fig. 1.35. The back of the cube is open.

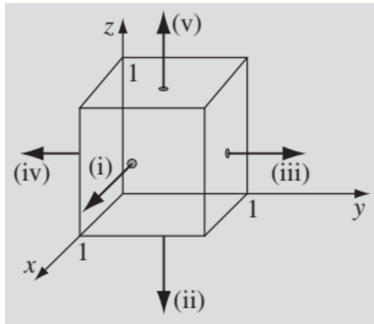

Fig. 1.35

Solution

Corollary 1 of Stokes's theorem says that the integral of a curl over a surface S,

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S},$$

depends only on the boundary curve of the surface, not on the surface itself. This will be tested here by first integrating the curl of $\mathbf{v} = \langle 0, 2xz + 3y^2, 4yz^2 \rangle$ over the surface consisting of the five labelled faces in Fig. 1.35 and then integrating the curl of $\mathbf{v}$ over the unlabelled sixth face lying in the yz-plane.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \iint_{\text{face (i)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} + \iint_{\text{face (ii)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} + \iint_{\text{face (iii)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \\
&\quad + \iint_{\text{face (iv)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} + \iint_{\text{face (v)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \\
&= \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (\hat{\mathbf{x}}\, dy\, dz)\Big|_{x=1} + \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (-\hat{\mathbf{z}}\, dx\, dy)\Big|_{z=0} \\
&\quad + \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (\hat{\mathbf{y}}\, dx\, dz)\Big|_{y=1} + \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (-\hat{\mathbf{y}}\, dx\, dz)\Big|_{y=0} \\
&\quad + \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (\hat{\mathbf{z}}\, dx\, dy)\Big|_{z=1}
\end{aligned}$$

Calculate the curl of $\mathbf{v}$.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} = \hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)$$

As a result, the surface integral of $\nabla \times \mathbf{v}$ over the five faces is

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (\hat{\mathbf{x}}\, dy\, dz)\Big|_{x=1} + \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (-\hat{\mathbf{z}}\, dx\, dy)\Big|_{z=0} \\
&\quad + \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (\hat{\mathbf{y}}\, dx\, dz)\Big|_{y=1} + \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (-\hat{\mathbf{y}}\, dx\, dz)\Big|_{y=0} \\
&\quad + \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (\hat{\mathbf{z}}\, dx\, dy)\Big|_{z=1} \\
&= \int_0^1 \int_0^1 (4z^2 - 2x)\Big|_{x=1} dy\, dz - \int_0^1 \int_0^1 (2z)\Big|_{z=0} dx\, dy \\
&\quad + \int_0^1 \int_0^1 (0)\Big|_{y=1} dx\, dz - \int_0^1 \int_0^1 (0)\Big|_{y=0} dx\, dz \\
&\quad + \int_0^1 \int_0^1 (2z)\Big|_{z=1} dx\, dy \\
&= \int_0^1 \int_0^1 (4z^2 - 2)\, dy\, dz + \int_0^1 \int_0^1 (2)\, dx\, dy \\
&= \left(-\frac{2}{3}\right) + (2) \\
&= \frac{4}{3}.
\end{aligned}$$

Now integrate $\nabla \times \mathbf{v}$ over the sixth unlabelled face. It's worth emphasizing here that the orientation of the area element is determined by the orientation of the boundary curve using the right-hand corkscrew rule, not by the direction of the outward normal as when using Gauss's theorem.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \iint_{\text{face (vi)}} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \int_0^1 \int_0^1 (\nabla \times \mathbf{v}) \cdot (\hat{\mathbf{x}}\, dy\, dz)\Big|_{x=0} \\
&= \int_0^1 \int_0^1 [\hat{\mathbf{x}}(4z^2 - 2x) + \hat{\mathbf{z}}(2z)] \cdot (\hat{\mathbf{x}}\, dy\, dz)\Big|_{x=0} \\
&= \int_0^1 \int_0^1 (4z^2 - 2x)\Big|_{x=0} dy\, dz \\
&= \int_0^1 \int_0^1 (4z^2)\, dy\, dz \\
&= \frac{4}{3}
\end{aligned}$$

Corollary 1 is verified.