Problem 1.39

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

(b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Solution

In spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis, the divergence of a vector function is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla \cdot \mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

Part (a)

If $\mathbf{v} = r^2\hat{\mathbf{r}}$ and D represents the sphere of radius R centered at the origin, then the left side evaluates to

$$\begin{aligned}
\iiint_D \nabla \cdot \mathbf{v}\, dV &= \int_0^\pi \int_0^{2\pi} \int_0^R \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^2)\right] r^2 \sin\theta\, dr\, d\phi\, d\theta \\
&= \int_0^\pi \int_0^{2\pi} \int_0^R \left[\frac{1}{r^2}(4r^3)\right] r^2 \sin\theta\, dr\, d\phi\, d\theta \\
&= 4\int_0^\pi \int_0^{2\pi} \int_0^R r^3 \sin\theta\, dr\, d\phi\, d\theta \\
&= 4\left(\int_0^R r^3\, dr\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^\pi \sin\theta\, d\theta\right) \\
&= 4\left(\frac{R^4}{4}\right)(2\pi)(2) \\
&= 4\pi R^4,
\end{aligned}$$

and the right side evaluates to

$$\begin{aligned}
\oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S} &= \int_0^\pi \int_0^{2\pi} (r^2\hat{\mathbf{r}})\Big|_{r=R} \cdot (\hat{\mathbf{r}}R^2 \sin\theta\, d\phi\, d\theta) \\
&= \int_0^\pi \int_0^{2\pi} (R^2\hat{\mathbf{r}}) \cdot (\hat{\mathbf{r}}R^2 \sin\theta\, d\phi\, d\theta) \\
&= R^4 \int_0^\pi \int_0^{2\pi} \sin\theta\, d\phi\, d\theta \\
&= 4\pi R^4.
\end{aligned}$$

Part (b)

If $\mathbf{v} = (1/r^2)\hat{\mathbf{r}}$ and D represents the sphere of radius R centered at the origin, then the left side evaluates to

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right)\right] r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \left[\frac{1}{r^2}(0)\right] r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$= 0,$$

and the right side evaluates to

$$\oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{r^2}\hat{\mathbf{r}}\right)\Big|_{r=R} \cdot (\hat{\mathbf{r}} R^2 \sin\theta\, d\phi\, d\theta)$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot (\hat{\mathbf{r}} R^2 \sin\theta\, d\phi\, d\theta)$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin\theta\, d\phi\, d\theta$$

$$= 4\pi.$$

Applying the formula for $\nabla \cdot \mathbf{v}$ leads to the incorrect answer due to the singularity at $r = 0$. There's a radial source at the origin and no sinks, so the volume integral has to be nonzero. Based on the divergence theorem, though, one can conclude that

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta(x)\delta(y)\delta(z) = 4\pi\delta(\mathbf{x}) = 4\pi\delta(\mathbf{r}) = 4\pi\delta^3(\mathbf{r}).$$

This way the volume integral gives the same answer.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \iiint_D 4\pi\delta(\mathbf{x})\, dV = 4\pi \left[\iiint_D \delta(\mathbf{x})\, dV\right] = 4\pi(1) = 4\pi$$