Problem 1.41

Compute the gradient and Laplacian of the scalar function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian coordinates and using Eq. 1.42. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from $(0,0,0)$ to $(0,0,2)$.

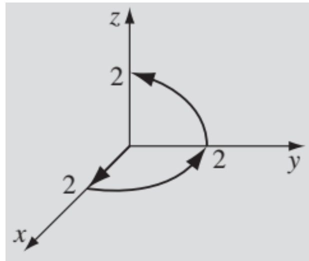

Fig. 1.41

Solution

In spherical coordinates (r,ϕ,θ), where θ is the angle from the polar axis, the gradient of a scalar function is

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}},$$

and the Laplacian of a scalar function is

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}.$$

For the given function, the gradient evaluates to

$$\begin{aligned}\nabla T &= \frac{\partial}{\partial r}[r(\cos\theta+\sin\theta\cos\phi)]\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}[r(\cos\theta+\sin\theta\cos\phi)]\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[r(\cos\theta+\sin\theta\cos\phi)]\hat{\boldsymbol{\phi}} \\ &= (\cos\theta+\sin\theta\cos\phi)\hat{\mathbf{r}} + \frac{1}{r}[r(-\sin\theta+\cos\theta\cos\phi)]\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}[r(-\sin\theta\sin\phi)]\hat{\boldsymbol{\phi}} \\ &= (\cos\theta+\sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta+\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} + (-\sin\phi)\hat{\boldsymbol{\phi}},\end{aligned}$$

the Laplacian evaluates to

$$\begin{aligned}\nabla^2 T &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}[r(\cos\theta+\sin\theta\cos\phi)]\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}[r(\cos\theta+\sin\theta\cos\phi)]\right\} \\ &\qquad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}[r(\cos\theta+\sin\theta\cos\phi)] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\cos\theta+\sin\theta\cos\phi)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\{\sin\theta[r(-\sin\theta+\cos\theta\cos\phi)]\} \\ &\qquad + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}[r(-\sin\theta\sin\phi)] \\ &= \frac{1}{r^2}[2r(\cos\theta+\sin\theta\cos\phi)] + \frac{1}{r^2\sin\theta}\{\cos\theta[r(-\sin\theta+\cos\theta\cos\phi)] + \sin\theta[r(-\cos\theta-\sin\theta\cos\phi)]\} \\ &\qquad + \frac{1}{r^2\sin^2\theta}[r(-\sin\theta\cos\phi)].\end{aligned}$$

Simplify the result.

$$\begin{aligned}
\nabla^2 T &= \frac{2}{r}(\cancel{\cos\theta} + \sin\theta\cos\phi) + \frac{1}{r^2\sin\theta}(\cancel{-2r\sin\theta\cos\theta} + r\cos^2\theta\cos\phi - r\sin^2\theta\cos\phi) + \frac{1}{r^2\sin^2\theta}(-r\sin\theta\cos\phi) \\
&= \cancel{\frac{2}{r}(\sin\theta\cos\phi)} + \frac{1}{r^2\sin\theta}(r\cos\phi - \cancel{r\sin^2\theta\cos\phi} - \cancel{r\sin^2\theta\cos\phi}) + \frac{1}{r^2\sin^2\theta}(-r\sin\theta\cos\phi) \\
&= \frac{\cos\phi}{r\sin\theta} - \frac{\cos\phi}{r\sin\theta} \\
&= 0
\end{aligned}$$

This result can be checked by converting the given function to Cartesian coordinates.

$$\begin{aligned}
T &= r(\cos\theta + \sin\theta\cos\phi) \\
&= r\cos\theta + r\sin\theta\cos\phi \\
&= z + x
\end{aligned}$$

Take the Laplacian of the function again, this time in Cartesian coordinates.

$$\begin{aligned}
\nabla^2 T &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}(z+x) + \frac{\partial^2}{\partial y^2}(z+x) + \frac{\partial^2}{\partial z^2}(z+x) \\
&= \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) \\
&= (0) + (0) + (0) \\
&= 0
\end{aligned}$$

The fundamental theorem for gradients implies that gradients are conservative vector fields: the line integral of a gradient is independent of the path taken.

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

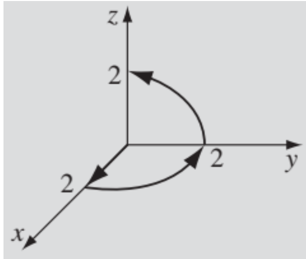

Fig. 1.41

For $T(x, y, z) = x + z$ and the path illustrated in Fig. 1.41, the right side evaluates to

$$\begin{aligned} T(\mathbf{b}) - T(\mathbf{a}) &= T(0,0,2) - T(0,0,0) \\ &= (0+2) - (0+0) \\ &= 2, \end{aligned}$$

and the left side evaluates to

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = \int_{\langle 0,0,0\rangle}^{\langle 2,0,0\rangle} (\nabla T) \cdot d\mathbf{l} + \int_{\langle 2,0,0\rangle}^{\langle 0,2,0\rangle} (\nabla T) \cdot d\mathbf{l} + \int_{\langle 0,2,0\rangle}^{\langle 0,0,2\rangle} (\nabla T) \cdot d\mathbf{l}.$$

Parameterize the three different segments.

$$\begin{aligned} &\langle 0,0,0\rangle \to \langle 2,0,0\rangle : && \mathbf{l}_1(t) = \langle t, 0, 0\rangle, && 0 \le t \le 2 \\ &\langle 2,0,0\rangle \to \langle 0,2,0\rangle : && \mathbf{l}_2(t) = 2\langle \cos t, \sin t, 0\rangle, && 0 \le t \le \frac{\pi}{2} \\ &\langle 0,2,0\rangle \to \langle 0,0,2\rangle : && \mathbf{l}_3(t) = 2\langle 0, \cos t, \sin t\rangle, && 0 \le t \le \frac{\pi}{2} \end{aligned}$$

Therefore, since $\nabla T = \langle \partial T/\partial x, \partial T/\partial y, \partial T/\partial z\rangle = \langle 1, 0, 1\rangle$,

$$\begin{aligned} \int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} &= \int_0^2 \nabla T(\mathbf{l}_1(t)) \cdot \mathbf{l}_1'(t)\, dt + \int_0^{\pi/2} \nabla T(\mathbf{l}_2(t)) \cdot \mathbf{l}_2'(t)\, dt + \int_0^{\pi/2} \nabla T(\mathbf{l}_3(t)) \cdot \mathbf{l}_3'(t)\, dt \\ &= \int_0^2 \langle 1,0,1\rangle \cdot \langle 1,0,0\rangle\, dt + \int_0^{\pi/2} \langle 1,0,1\rangle \cdot 2\langle -\sin t, \cos t, 0\rangle\, dt + \int_0^{\pi/2} \langle 1,0,1\rangle \cdot 2\langle 0, -\sin t, \cos t\rangle\, dt \\ &= \int_0^2 (1)\, dt + \int_0^{\pi/2} (-2\sin t)\, dt + \int_0^{\pi/2} (2\cos t)\, dt \\ &= (2) + (-2) + (2) \\ &= 2. \end{aligned}$$