Problem 1.42

Express the cylindrical unit vectors $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.75). "Invert" your formulas to get $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ (and ϕ).

Solution

Eq. 1.74 gives the formulas to switch from Cartesian coordinates (x, y, z) into cylindrical coordinates (s, ϕ, z).

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \tag{1.62}$$

The horizontal position vector from the z-axis $(0, 0, z)$ to the point (x, y, z) is written as

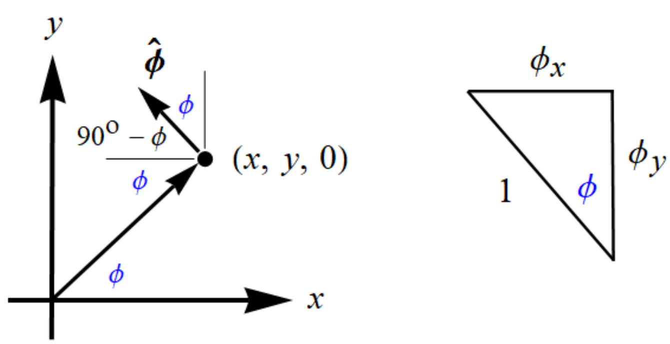
$$\mathbf{s} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

$$s\hat{\mathbf{s}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}.$$

Divide both sides by s to get the radial unit vector.

$$\begin{aligned} \hat{\mathbf{s}} &= \frac{x}{s}\hat{\mathbf{x}} + \frac{y}{s}\hat{\mathbf{y}} \\ &= \frac{s\cos\phi}{s}\hat{\mathbf{x}} + \frac{s\sin\phi}{s}\hat{\mathbf{y}} \\ &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \end{aligned}$$

In order to get a formula for $\hat{\boldsymbol{\phi}}$, consider a point in the xy-plane for simplicity and use trigonometry. The triangle on the right is of the magnitude of $\hat{\boldsymbol{\phi}}$ and the magnitudes of the components along the x- and y-axes.



The x-component of $\hat{\boldsymbol{\phi}}$ is $-\sin\phi$, and the y-component of $\hat{\boldsymbol{\phi}}$ is $\cos\phi$:

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$$

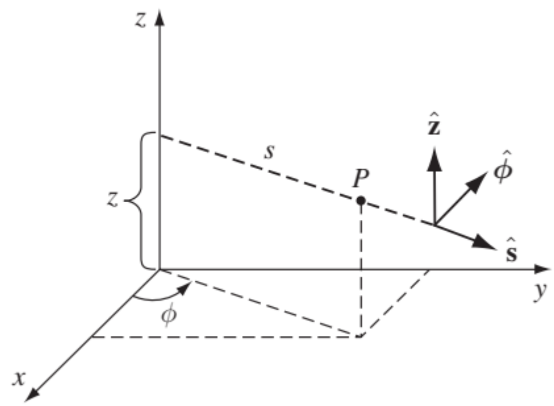


Fig. 1.42

By the right-hand corkscrew rule, the remaining unit vector $\hat{\mathbf{z}}$ is given by

$$\begin{aligned}\hat{\mathbf{z}} &= \hat{\mathbf{s}} \times \hat{\boldsymbol{\phi}} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\ &= 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + (\cos^2\phi + \sin^2\phi)\hat{\mathbf{z}} \\ &= \hat{\mathbf{z}}.\end{aligned}$$

To summarize,

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}.$$

In order to get the formulas for $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, write this as a matrix equation.

$$\begin{bmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}$$

Consequently,

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{bmatrix}.$$

Find the inverse of the matrix.

$$\left[\begin{array}{ccc|ccc} \cos\phi & \sin\phi & 0 & 1 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{1}{\cos\phi} & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{1}{\cos\phi} & 0 & 0 \\ 0 & \cos\phi + \frac{\sin^2\phi}{\cos\phi} & 0 & \frac{\sin\phi}{\cos\phi} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{1}{\cos\phi} & 0 & 0 \\ 0 & \frac{1}{\cos\phi} & 0 & \frac{\sin\phi}{\cos\phi} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{\sin\phi}{\cos\phi} & 0 & \frac{1}{\cos\phi} & 0 & 0 \\ 0 & 1 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{\cos\phi} - \frac{\sin^2\phi}{\cos\phi} & -\sin\phi & 0 \\ 0 & 1 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 1 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The inverse matrix is then

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which means

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{bmatrix}.$$

Therefore,

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}.$$