Problem 1.51

For Theorem 1, show that (d) ⇒ (a), (a) ⇒ (c), (c) ⇒ (b), (b) ⇒ (c), and (c) ⇒ (a).

Solution

Theorem 1 says that the following conditions are equivalent.

(a) $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path for any given end points.

(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

(d) $\mathbf{F}$ is the gradient of some scalar function: $\mathbf{F} = -\nabla V$.

(d) ⇒ (a)

Assume that $\mathbf{F}$ is the gradient of some scalar function: $\mathbf{F} = -\nabla V$. Show that $\nabla \times \mathbf{F} = \mathbf{0}$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times (-\nabla V) \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[-\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) V\right] \\
&= -\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial V}{\partial x_j}\right) \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) \\
&= -\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_i}\right) \qquad \text{(let } i \text{ be } j \text{ and let } j \text{ be } i\text{)} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_i}\right) \qquad \text{(limits are constant, so interchange sums)} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) \qquad \text{(use Clairaut's theorem)} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k (-\varepsilon_{ijk}) \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}\right) = \mathbf{0}
\end{aligned}$$

(a) ⇒ (c)

Assume that $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere and show that $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

$$\nabla \times \mathbf{F} = \mathbf{0}$$

Integrate both sides over any open surface S with boundary line, bdy S.

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S}$$

Use Stokes's theorem on the left and evaluate the integral on the right.

$$\oint_{\text{bdy } S} \mathbf{F} \cdot d\mathbf{l} = 0$$

(c) ⇒ (b)

Assume that $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ and show that $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path for any given end points.

$$\oint_{\text{bdy } S} \mathbf{F} \cdot d\mathbf{l} = 0$$

By the fundamental theorem for gradients, there exists a potential function T such that $\mathbf{F} = \nabla T$. Gradients are known to be conservative vector fields. Therefore,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = \int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l}$$

is independent of path for any given end points.

(b) ⇒ (c)

Assume that

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$$

is independent of path for any given end points. That means $\mathbf{F}$ is a conservative vector field. Integrate $\mathbf{F}$ from $\mathbf{a}$ to $\mathbf{b}$ and then back to $\mathbf{a}$.

$$\begin{aligned}
\oint \mathbf{F} \cdot d\mathbf{l} &= \int_{\mathbf{a}}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l} \\
&= \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} + \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l} \\
&= \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} \\
&= 0
\end{aligned}$$

(c) ⇒ (a)

Assume that

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = 0$$

for any closed loop C. Apply Stokes's theorem to turn this line integral into a surface integral.

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$$

Since this is true for any surface S, $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere.