Problem 1.56

Compute the line integral of

$$\mathbf{v} = 6\hat{\mathbf{x}} + yz^2\hat{\mathbf{y}} + (3y+z)\hat{\mathbf{z}}$$

along the triangular path shown in Fig. 1.49. Check your answer using Stokes' theorem. [*Answer:* 8/3.]

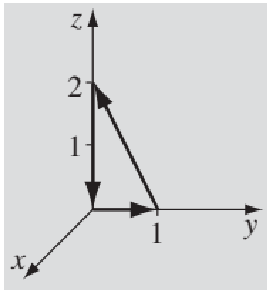


Fig. 1.49

Solution

Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Let S be the triangular area enclosed by the triangular path in Fig. 1.49. Evaluate the left side, noting that the direction of the area element is given by the right-hand corkscrew rule.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \iint_S \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6 & yz^2 & 3y+z \end{vmatrix} \cdot d\mathbf{S} \\
&= \iint_S \left\{ \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(3y+z) - \frac{\partial}{\partial z}(yz^2) \right] - \hat{\mathbf{y}} \left[\frac{\partial}{\partial x}(3y+z) - \frac{\partial}{\partial z}(6) \right] \right. \\
&\qquad \left. + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial y}(6) \right] \right\} \cdot d\mathbf{S} \\
&= \iint_S \{ \hat{\mathbf{x}}[(3) - (2yz)] - \hat{\mathbf{y}}[(0) - (0)] + \hat{\mathbf{z}}[(0) - (0)] \} \cdot d\mathbf{S} \\
&= \int_0^1 \int_0^{2-2y} (3 - 2yz)\hat{\mathbf{x}} \cdot (\hat{\mathbf{x}}\, dz\, dy) \\
&= \int_0^1 \int_0^{2-2y} (3 - 2yz)\, dz\, dy
\end{aligned}$$

Continue the simplification.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \int_0^1 (3z - yz^2)\Big|_0^{2-2y} dy$$

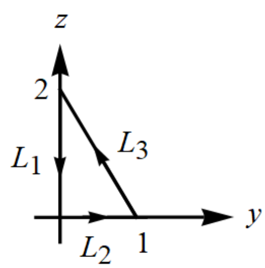
$$= \int_0^1 [3(2-2y) - y(2-2y)^2]\, dy$$

$$= \int_0^1 (6 - 10y + 8y^2 - 4y^3)\, dy$$

$$= \left(6y - 5y^2 + \frac{8}{3}y^3 - y^4\right)\Big|_0^1$$

$$= \frac{8}{3}$$

Label the different segments of the path



and parameterize them.

$$\langle 0,0,2\rangle \to \langle 0,0,0\rangle : \qquad \mathbf{l}_1(t) = \langle 0,0,2-t\rangle, \qquad 0 \le t \le 2$$

$$\langle 0,0,0\rangle \to \langle 0,1,0\rangle : \qquad \mathbf{l}_2(t) = \langle 0,t,0\rangle, \qquad 0 \le t \le 1$$

$$\langle 0,1,0\rangle \to \langle 0,0,2\rangle : \qquad \mathbf{l}_3(t) = \langle 0,1-t,2t\rangle, \qquad 0 \le t \le 1$$

As a result, the right side of Stokes's theorem becomes

$$\oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l} = \int_{L_1} \mathbf{v} \cdot d\mathbf{l} + \int_{L_2} \mathbf{v} \cdot d\mathbf{l} + \int_{L_3} \mathbf{v} \cdot d\mathbf{l}$$

$$= \int_0^2 \mathbf{v}(\mathbf{l}_1(t)) \cdot \mathbf{l}_1'(t)\, dt + \int_0^1 \mathbf{v}(\mathbf{l}_2(t)) \cdot \mathbf{l}_2'(t)\, dt + \int_0^1 \mathbf{v}(\mathbf{l}_3(t)) \cdot \mathbf{l}_3'(t)\, dt$$

$$= \int_0^2 \langle 6, (0)(2-t)^2, 3(0) + (2-t)\rangle \cdot \langle 0,0,-1\rangle\, dt + \int_0^1 \langle 6, (t)(0)^2, 3(t) + (0)\rangle \cdot \langle 0,1,0\rangle\, dt$$

$$+ \int_0^1 \langle 6, (1-t)(2t)^2, 3(1-t) + (2t)\rangle \cdot \langle 0,-1,2\rangle\, dt.$$

Continue the simplification.

$$\begin{aligned}
\oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l} &= \int_0^2 (t-2)\,dt + \int_0^1 (0)\,dt + \int_0^1 \{-(1-t)(2t)^2 + 2[3(1-t) + (2t)]\}\,dt \\
&= \int_0^2 (t-2)\,dt + \int_0^1 (6 - 2t - 4t^2 + 4t^3)\,dt \\
&= \left(\frac{t^2}{2} - 2t\right)\Big|_0^2 + \left(6t - t^2 - \frac{4}{3}t^3 + t^4\right)\Big|_0^1 \\
&= (-2) + \left(\frac{14}{3}\right) \\
&= \frac{8}{3}
\end{aligned}$$