# Problem 1.57

Compute the line integral of

$$\mathbf{v} = (r\cos^2\theta)\hat{\mathbf{r}} - (r\cos\theta\sin\theta)\hat{\boldsymbol{\theta}} + 3r\hat{\boldsymbol{\phi}}$$

around the path shown in Fig. 1.50 (the points are labeled by their Cartesian coordinates). Do it either in cylindrical or in spherical coordinates. Check your answer, using Stokes' theorem. [*Answer:* $3\pi/2$.]

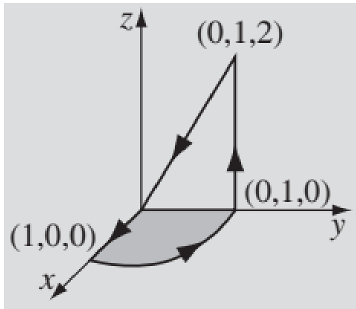


**Fig. 1.50**

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### Solution

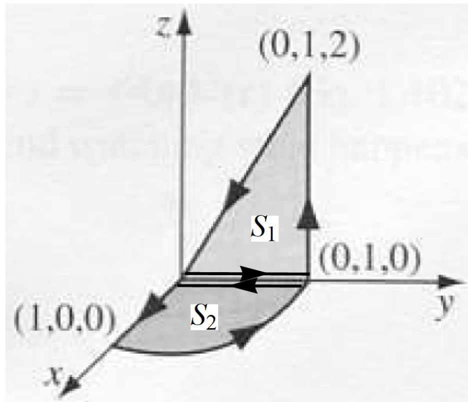
Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

The curl of the given function is

$$\begin{aligned}
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\phi\sin\theta) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}[(3r)\sin\theta] - \frac{\partial}{\partial\phi}(-r\cos\theta\sin\theta)\right\}\hat{\mathbf{r}} + \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\cos^2\theta) - \frac{\partial}{\partial r}[r(3r)]\right\}\hat{\boldsymbol{\theta}} \\
&\qquad + \frac{1}{r}\left\{\frac{\partial}{\partial r}[r(-r\cos\theta\sin\theta)] - \frac{\partial}{\partial\theta}(r\cos^2\theta)\right\}\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r\sin\theta}\left\{[(3r)\cos\theta] - (0)\right\}\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}(0) - (6r)\right]\hat{\boldsymbol{\theta}} \\
&\qquad + \frac{1}{r}\left\{(-2r\cos\theta\sin\theta) - [2r\cos\theta(-\sin\theta)]\right\}\hat{\boldsymbol{\phi}} \\
&= 3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}} + 0\,\hat{\boldsymbol{\phi}}.
\end{aligned}$$

Let $S$ be the gray area enclosed by the triangular and circular paths in Fig. 1.50.




Evaluate the left side of Stokes's theorem by splitting up the double integral over each shape, noting that the direction of each area element is given by the right-hand corkscrew rule.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \iint_{S_1} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} + \iint_{S_2} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \\
&= \int_0^1 \int_0^{2y} (3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}})\Big|_{\phi=\pi/2} \cdot (-\hat{\boldsymbol{\phi}}\,dz\,dy) + \int_0^{\pi/2} \int_0^1 (3\cot\theta\,\hat{\mathbf{r}} - 6\,\hat{\boldsymbol{\theta}})\Big|_{\theta=\pi/2} \cdot (-\hat{\boldsymbol{\theta}}\,r\,dr\,d\phi) \\
&= \int_0^1 \int_0^{2y} (0)\,dz\,dy + \int_0^{\pi/2} \int_0^1 (6r)\,dr\,d\phi \\
&= 6\left(\int_0^{\pi/2} d\phi\right)\left(\int_0^1 r\,dr\right) \\
&= 6\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{3\pi}{2}
\end{aligned}$$

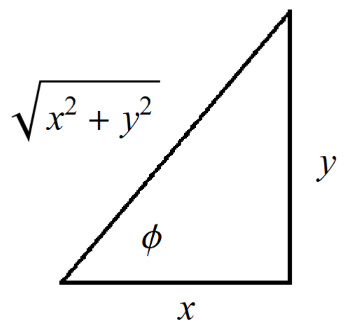
Rewrite the given vector in Cartesian coordinates.

$$\begin{aligned}
\mathbf{v} &= (r\cos^2\theta)\,\hat{\mathbf{r}} - (r\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}} + 3r\,\hat{\boldsymbol{\phi}} \\
&= (r\cos^2\theta)(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\
&\qquad - (r\cos\theta\sin\theta)(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}) \\
&\qquad\qquad + 3r(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \\
&= (\cancel{r\sin\theta\cos^2\theta\cos\phi} - \cancel{r\sin\theta\cos^2\theta\cos\phi} - 3r\sin\phi)\hat{\mathbf{x}} \\
&\qquad + (\cancel{r\sin\theta\cos^2\theta\sin\phi} - \cancel{r\sin\theta\cos^2\theta\sin\phi} + 3r\cos\phi)\hat{\mathbf{y}} \\
&\qquad\qquad + (r\cos^3\theta + r\sin^2\theta\cos\theta)\hat{\mathbf{z}}
\end{aligned}$$

Simplify the result.

$$\begin{aligned}\mathbf{v} &= -3r\sin\phi\,\hat{\mathbf{x}} + 3r\cos\phi\,\hat{\mathbf{y}} + [r(1-\sin^2\theta)\cos\theta + r\sin^2\theta\cos\theta]\hat{\mathbf{z}} \\ &= -3r\sin\phi\,\hat{\mathbf{x}} + 3r\cos\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} \\ &= -3r\sin\phi\,\hat{\mathbf{x}} + 3r\cos\phi\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\end{aligned}$$

Use the fact that $\phi = \tan^{-1}(y/x)$ to draw the implied right triangle and determine the sine and cosine.

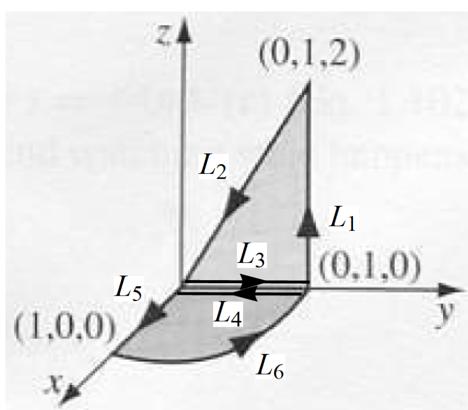



$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

Consequently, in Cartesian coordinates,

$$\mathbf{v} = -3\sqrt{x^2+y^2+z^2}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\hat{\mathbf{x}} + 3\sqrt{x^2+y^2+z^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$

Now label each of the line segments



and parameterize them.

$$\begin{aligned}
&\langle 0,1,0\rangle \to \langle 0,1,2\rangle : && \mathbf{l}_1 = \langle 0,1,t\rangle, && 0 \le t \le 2 \\
&\langle 0,1,2\rangle \to \langle 0,0,0\rangle : && \mathbf{l}_2 = \langle 0,1-t,2-2t\rangle, && 0 \le t \le 1 \\
&\langle 0,0,0\rangle \to \langle 0,1,0\rangle : && \mathbf{l}_3 = \langle 0,t,0\rangle, && 0 \le t \le 1 \\
&\langle 0,1,0\rangle \to \langle 0,0,0\rangle : && \mathbf{l}_4 = \langle 0,1-t,0\rangle, && 0 \le t \le 1 \\
&\langle 0,0,0\rangle \to \langle 1,0,0\rangle : && \mathbf{l}_5 = \langle t,0,0\rangle, && 0 \le t \le 1 \\
&\langle 1,0,0\rangle \to \langle 0,1,0\rangle : && \mathbf{l}_6 = \langle \cos t,\sin t,0\rangle, && 0 \le t \le \frac{\pi}{2}
\end{aligned}$$

Finally, evaluate the right side of Stokes's theorem.

$$\begin{aligned}
\oint_{\text{bdy } S} \mathbf{v}\cdot d\mathbf{l} &= \int_{L_1} \mathbf{v}\cdot d\mathbf{l} + \int_{L_2} \mathbf{v}\cdot d\mathbf{l} + \int_{L_3} \mathbf{v}\cdot d\mathbf{l} + \int_{L_4} \mathbf{v}\cdot d\mathbf{l} + \int_{L_5} \mathbf{v}\cdot d\mathbf{l} + \int_{L_6} \mathbf{v}\cdot d\mathbf{l} \\
&= \int_0^2 \mathbf{v}(\mathbf{l}_1(t))\cdot \mathbf{l}_1'(t)\,dt + \int_0^1 \mathbf{v}(\mathbf{l}_2(t))\cdot \mathbf{l}_2'(t)\,dt + \int_0^1 \mathbf{v}(\mathbf{l}_3(t))\cdot \mathbf{l}_3'(t)\,dt \\
&\qquad + \int_0^1 \mathbf{v}(\mathbf{l}_4(t))\cdot \mathbf{l}_4'(t)\,dt + \int_0^1 \mathbf{v}(\mathbf{l}_5(t))\cdot \mathbf{l}_5'(t)\,dt + \int_0^{\pi/2} \mathbf{v}(\mathbf{l}_6(t))\cdot \mathbf{l}_6'(t)\,dt \\
&= \int_0^2 \langle -3\sqrt{1+t^2},0,t\rangle \cdot \langle 0,0,1\rangle\,dt + \int_0^1 \langle 3\sqrt{5}(t-1),0,2-2t\rangle \cdot \langle 0,-1,-2\rangle\,dt \\
&\qquad + \int_0^1 \langle -3t,0,0\rangle \cdot \langle 0,1,0\rangle\,dt + \int_0^1 \langle 3(t-1),0,0\rangle \cdot \langle 0,-1,0\rangle\,dt \\
&\qquad\qquad + \int_0^1 \langle 0,3t,0\rangle \cdot \langle 1,0,0\rangle\,dt + \int_0^{\pi/2} \langle -3\sin t,3\cos t,0\rangle \cdot \langle -\sin t,\cos t,0\rangle\,dt \\
&= \int_0^2 (t)\,dt + \int_0^1 [-2(2-2t)]\,dt + \int_0^1 (0)\,dt + \int_0^1 (0)\,dt + \int_0^1 (0)\,dt + \int_0^{\pi/2} (3\sin^2 t + 3\cos^2 t)\,dt \\
&= \int_0^2 t\,dt + 4\int_0^1 (t-1)\,dt + 3\int_0^{\pi/2} dt \\
&= (2) + 4\left(-\frac{1}{2}\right) + 3\left(\frac{\pi}{2}\right) \\
&= \frac{3\pi}{2}
\end{aligned}$$