Problem 1.58

Check Stokes' theorem for the function $\mathbf{v} = y\hat{\mathbf{z}}$, using the triangular surface shown in Fig. 1.51. [*Answer:* a^2.]

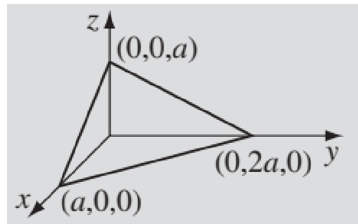


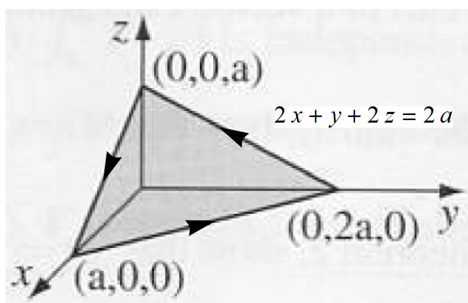
Fig. 1.51

Solution

Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

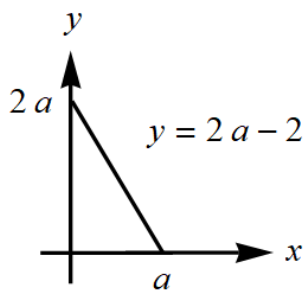
$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Let S be the positively oriented gray area enclosed by the triangle in Fig. 1.51.



Let $f(x, y, z) = 2x + y + 2z$ and $z = g(x, y) = a - x - y/2$. Evaluate the left side of Stokes's theorem, noting that the direction of the area element is given by the right-hand corkscrew rule.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \iint_S \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y \end{vmatrix} \cdot (\hat{\mathbf{n}}\, dS) \\
&= \iint_S \left\{ \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(0) \right] - \hat{\mathbf{y}} \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(0) \right] + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0) \right] \right\} \cdot \frac{\nabla f}{|\nabla f|}\, dS \\
&= \iint_S \{ \hat{\mathbf{x}} [(1) - (0)] - \hat{\mathbf{y}} [(0) - (0)] + \hat{\mathbf{z}} [(0) - (0)] \} \cdot \frac{\langle 2, 1, 2 \rangle}{\sqrt{2^2 + 1^2 + 2^2}}\, dS \\
&= \iint_A \langle 1, 0, 0 \rangle \cdot \frac{\langle 2, 1, 2 \rangle}{\sqrt{2^2 + 1^2 + 2^2}} \sqrt{1 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2}\, dA
\end{aligned}$$



Here A is the projection of S onto the xy-plane.
As a result,

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \int_0^a \int_0^{2a-2x} \langle 1,0,0\rangle \cdot \frac{\langle 2,1,2\rangle}{\sqrt{2^2+1^2+2^2}} \sqrt{1+(-1)^2+\left(-\frac{1}{2}\right)^2}\, dy\, dx$$

$$= \int_0^a \int_0^{2a-2x} \frac{2}{\sqrt{9}} \sqrt{\frac{9}{4}}\, dy\, dx$$

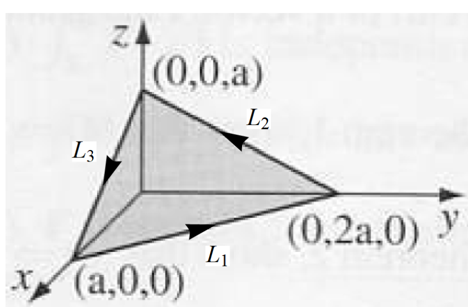
$$= \int_0^a \int_0^{2a-2x} dy\, dx$$

$$= \int_0^a (2a-2x)\, dx$$

$$= (2ax - x^2)\Big|_0^a$$

$$= a^2.$$

Label each of the line segments



and parameterize them.

$$\langle a,0,0\rangle \to \langle 0,2a,0\rangle : \qquad \mathbf{l}_1 = \langle a-t, 2t, 0\rangle, \qquad 0 \le t \le a$$

$$\langle 0,2a,0\rangle \to \langle 0,0,a\rangle : \qquad \mathbf{l}_2 = \langle 0, 2a-2t, t\rangle, \qquad 0 \le t \le a$$

$$\langle 0,0,a\rangle \to \langle a,0,0\rangle : \qquad \mathbf{l}_3 = \langle t, 0, a-t\rangle, \qquad 0 \le t \le a$$

Evaluate the right side of Stokes's theorem.

$$\begin{aligned}
\oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l} &= \int_{L_1} \mathbf{v} \cdot d\mathbf{l} + \int_{L_2} \mathbf{v} \cdot d\mathbf{l} + \int_{L_3} \mathbf{v} \cdot d\mathbf{l} \\
&= \int_0^a \mathbf{v}(\mathbf{l}_1(t)) \cdot \mathbf{l}_1'(t)\, dt + \int_0^a \mathbf{v}(\mathbf{l}_2(t)) \cdot \mathbf{l}_2'(t)\, dt + \int_0^a \mathbf{v}(\mathbf{l}_3(t)) \cdot \mathbf{l}_3'(t)\, dt \\
&= \int_0^a \langle 0, 0, 2t \rangle \cdot \langle -1, 2, 0 \rangle\, dt + \int_0^a \langle 0, 0, 2a - 2t \rangle \cdot \langle 0, -2, 1 \rangle\, dt + \int_0^a \langle 0, 0, 0 \rangle \cdot \langle 1, 0, -1 \rangle\, dt \\
&= \int_0^a (0)\, dt + \int_0^a (2a - 2t)\, dt + \int_0^a (0)\, dt \\
&= (0) + (2at - t^2)\Big|_0^a + (0) \\
&= a^2
\end{aligned}$$