Problem 1.6

Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}.$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

Solution

Use the BAC-CAB rule to simplify each of the expressions in square brackets.

$$\begin{aligned}[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= [\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})] \\ &\quad + [\mathbf{C}(\mathbf{B} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})] \\ &\quad + [\mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{C} \cdot \mathbf{A})]\end{aligned}$$

Use the fact that the dot product is commutative.

$$\begin{aligned}[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \cancel{\mathbf{C}(\mathbf{A} \cdot \mathbf{B})} \\ &\quad + \cancel{\mathbf{C}(\mathbf{A} \cdot \mathbf{B})} - \cancel{\mathbf{A}(\mathbf{B} \cdot \mathbf{C})} \\ &\quad + \cancel{\mathbf{A}(\mathbf{B} \cdot \mathbf{C})} - \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})}\end{aligned}$$

Therefore,

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}.$$

Bring $\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$ to the right side.

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] = -[\mathbf{C} \times (\mathbf{A} \times \mathbf{B})]$$

Use the minus sign to switch the order.

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] = [(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}]$$

We see that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ if and only if $\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{0}$, that is, if one of the following conditions is satisfied.

1. $\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C}$: $\mathbf{B} \cdot \mathbf{A} = 0$ and $\mathbf{B} \cdot \mathbf{C} = 0$.
2. $\mathbf{A}$ and $\mathbf{C}$ are parallel: $\mathbf{C} = \lambda \mathbf{A}$, where λ is a real constant.
3. $\mathbf{A}$, $\mathbf{B}$, or $\mathbf{C}$ are the zero vector: $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{C} = \mathbf{0}$.