Problem 1.60

Here are two cute checks of the fundamental theorems:

(a) Combine Corollary 2 to the gradient theorem with Stokes' theorem ($\mathbf{v} = \nabla T$ in this case). Show that the result is consistent with what you already knew about second derivatives.

(b) Combine Corollary 2 to Stokes' theorem with the divergence theorem. Show that the result is consistent with what you already knew.

Solution

Part (a)

Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Suppose that $\mathbf{v}$ is the gradient of a scalar function: $\mathbf{v} = \nabla u$.

$$\iint_S [\nabla \times (\nabla u)] \cdot d\mathbf{S} = \oint_{\text{bdy } S} (\nabla u) \cdot d\mathbf{l}$$

The right side is zero by Corollary 2 to the fundamental theorem for gradients (any gradient is a conservative vector field, so the closed loop integral of one is zero), and the left side is zero by Identity 10 (the curl of any gradient is zero).

$$\iint_S (\mathbf{0}) \cdot d\mathbf{S} = \int_{\mathbf{a}}^{\mathbf{a}} (\nabla u) \cdot d\mathbf{l}$$

$$0 = 0$$

Part (b)

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla \cdot \mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

Suppose that $\mathbf{v}$ is the curl of another vector function: $\mathbf{v} = \nabla \times \mathbf{u}$.

$$\iiint_D \nabla \cdot (\nabla \times \mathbf{u})\, dV = \oiint_{\text{bdy } D} (\nabla \times \mathbf{u}) \cdot d\mathbf{S}$$

The right side is zero by Corollary 2 to Stokes's theorem (the boundary line for a closed surface is a point), and the left side is zero by Identity 9 (the divergence of any curl is zero).

$$\iiint_D (0)\, dV = \int_0^0 \mathbf{u}(\mathbf{l}(t)) \cdot \mathbf{l}'(t)\, dt$$

$$0 = 0$$