Problem 1.61

Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show the following:

(a) $\int_{\mathcal{V}}(\nabla T)\,d\tau = \oint_{\mathcal{S}} T\,d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector, in the divergence theorem; use the product rules.]

(b) $\int_{\mathcal{V}}(\nabla \times \mathbf{v})\,d\tau = -\oint_{\mathcal{S}} \mathbf{v} \times d\mathbf{a}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v} \times \mathbf{c})$ in the divergence theorem.]

(c) $\int_{\mathcal{V}}[T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\,d\tau = \oint_{\mathcal{S}}(T\nabla U)\cdot d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = T\nabla U$ in the divergence theorem.]

(d) $\int_{\mathcal{V}}(T\nabla^2 U - U\nabla^2 T)\,d\tau = \oint_{\mathcal{S}}(T\nabla U - U\nabla T)\cdot d\mathbf{a}$. [*Comment:* This is sometimes called **Green's second identity**; it follows from (c), which is known as **Green's identity**.]

(e) $\int_{\mathcal{S}} \nabla T \times d\mathbf{a} = -\oint_{\mathcal{P}} T\,d\mathbf{l}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$ in Stokes' theorem.]

(f) $\int_{\mathcal{S}}[(d\mathbf{a} \times \nabla) \times \mathbf{v}] = -\oint_{\mathcal{P}} \mathbf{v} \times d\mathbf{l}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v} \times \mathbf{c})$ in Stokes' theorem.]

[TYPO: This is Green's first identity.]

Solution

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla \cdot \mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla \cdot \mathbf{v}\,dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

On the other hand, Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Part (a)

Begin with the divergence theorem, let $\mathbf{v} = \mathbf{c}T$ in which $\mathbf{c}$ is constant, and use Identity 5 from inside the front cover.

$$\iiint_D \nabla \cdot \mathbf{v}\,dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

$$\iiint_D \nabla \cdot (\mathbf{c}T)\,dV = \oiint_{\text{bdy } D} (\mathbf{c}T) \cdot d\mathbf{S}$$

$$\iiint_D [T\underbrace{(\nabla \cdot \mathbf{c})}_{=\,0} + \mathbf{c}\cdot(\nabla T)]\,dV = \mathbf{c} \cdot \oiint_{\text{bdy } D} T\,d\mathbf{S}$$

$$\mathbf{c} \cdot \iiint_D \nabla T\,dV = \mathbf{c} \cdot \oiint_{\text{bdy } D} T\,d\mathbf{S}$$

Therefore,

$$\iiint_D \nabla T\, dV = \oiint_{\text{bdy } D} T\, d\mathbf{S}.$$

Part (b)

Begin with the divergence theorem, let $\mathbf{v} = \mathbf{u} \times \mathbf{c}$ in which $\mathbf{c}$ is constant, use Identity 6 on the left, and use Identity 1 on the right.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

$$\iiint_D \nabla \cdot (\mathbf{u} \times \mathbf{c})\, dV = \oiint_{\text{bdy } D} (\mathbf{u} \times \mathbf{c}) \cdot d\mathbf{S}$$

$$\iiint_D [\mathbf{c} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot \underbrace{(\nabla \times \mathbf{c})}_{=\mathbf{0}}]\, dV = \oiint_{\text{bdy } D} \mathbf{c} \cdot (d\mathbf{S} \times \mathbf{u})$$

$$\mathbf{c} \cdot \iiint_D (\nabla \times \mathbf{u})\, dV = \mathbf{c} \cdot \oiint_{\text{bdy } D} (-\mathbf{u} \times d\mathbf{S})$$

Therefore,

$$\iiint_D (\nabla \times \mathbf{u})\, dV = -\oiint_{\text{bdy } D} \mathbf{u} \times d\mathbf{S}.$$

Part (c)

Begin with the divergence theorem, let $\mathbf{v} = T\nabla U$ in which T and U are any two scalar differentiable functions defined in the volume D, and use Identity 5 on the left side.

$$\iiint_D \nabla \cdot \mathbf{v}\, dV = \oiint_{\text{bdy } D} \mathbf{v} \cdot d\mathbf{S}$$

$$\iiint_D \nabla \cdot (T\nabla U)\, dV = \oiint_{\text{bdy } D} (T\nabla U) \cdot d\mathbf{S}$$

$$\iiint_D [T(\nabla \cdot \nabla U) + (\nabla U) \cdot (\nabla T)]\, dV = \oiint_{\text{bdy } D} (T\nabla U) \cdot d\mathbf{S}$$

Therefore,

$$\iiint_D [T\nabla^2 U + (\nabla T) \cdot (\nabla U)]\, dV = \oiint_{\text{bdy } D} (T\nabla U) \cdot d\mathbf{S}.$$

This is Green's first identity.

Part (d)

Green's first identity states that for any two functions, U and T,

$$\iiint_D [T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\,dV = \oiint_{\text{bdy } D} (T\nabla U)\cdot d\mathbf{S}. \tag{1}$$

It holds even if T and U are interchanged.

$$\iiint_D [U\nabla^2 T + (\nabla U)\cdot(\nabla T)]\,dV = \oiint_{\text{bdy } D} (U\nabla T)\cdot d\mathbf{S} \tag{2}$$

Subtract the respective sides of equation (2) from those of equation (1).

$$\iiint_D [T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\,dV - \iiint_D [U\nabla^2 T + (\nabla U)\cdot(\nabla T)]\,dV = \oiint_{\text{bdy } D} (T\nabla U)\cdot d\mathbf{S} - \oiint_{\text{bdy } D} (U\nabla T)\cdot d\mathbf{S}$$

$$\iiint_D [T\nabla^2 U + \cancel{(\nabla T)\cdot(\nabla U)} - U\nabla^2 T - \cancel{(\nabla U)\cdot(\nabla T)}]\,dV = \oiint_{\text{bdy } D} (T\nabla U - U\nabla T)\cdot d\mathbf{S}$$

Therefore,

$$\iiint_D (T\nabla^2 U - U\nabla^2 T)\,dV = \oiint_{\text{bdy } D} (T\nabla U - U\nabla T)\cdot d\mathbf{S}.$$

This is Green's second identity.

Part (e)

Begin with Stokes's theorem, let $\mathbf{v} = \mathbf{c}T$ in which $\mathbf{c}$ is constant, and use Identity 7 followed by Identity 1 on the left side.

$$\iint_S (\nabla \times \mathbf{v})\cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v}\cdot d\mathbf{l}$$

$$\iint_S [\nabla \times (\mathbf{c}T)]\cdot d\mathbf{S} = \oint_{\text{bdy } S} (\mathbf{c}T)\cdot d\mathbf{l}$$

$$\iint_S [T\underbrace{(\nabla \times \mathbf{c})}_{=\mathbf{0}} - \mathbf{c}\times(\nabla T)]\cdot d\mathbf{S} = \mathbf{c}\cdot \oint_{\text{bdy } S} T\,d\mathbf{l}$$

$$-\iint_S \mathbf{c}\cdot(\nabla T \times d\mathbf{S}) = \mathbf{c}\cdot \oint_{\text{bdy } S} T\,d\mathbf{l}$$

$$\mathbf{c}\cdot \iint_S (\nabla T \times d\mathbf{S}) = \mathbf{c}\cdot \oint_{\text{bdy } S} (-T)\,d\mathbf{l}$$

Therefore,

$$\iint_S \nabla T \times d\mathbf{S} = -\oint_{\text{bdy } S} T\, d\mathbf{l}.$$

Part (f)

Begin with Stokes's theorem.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Let $\mathbf{v} = \mathbf{u} \times \mathbf{c}$.

$$\iint_S [\nabla \times (\mathbf{u} \times \mathbf{c})] \cdot d\mathbf{S} = \oint_{\text{bdy } S} (\mathbf{u} \times \mathbf{c}) \cdot d\mathbf{l}$$

Use Identity 1 inside the front cover to rewrite the integral on the right side.

$$\begin{aligned}\iint_S [\nabla \times (\mathbf{u} \times \mathbf{c})] \cdot d\mathbf{S} &= \oint_{\text{bdy } S} (d\mathbf{l} \times \mathbf{u}) \cdot \mathbf{c} \\ &= \oint_{\text{bdy } S} (-\mathbf{u} \times d\mathbf{l}) \cdot \mathbf{c} \qquad (3)\end{aligned}$$

Now rewrite the integrand on the left side.

$$\begin{aligned}[\nabla \times (\mathbf{u} \times \mathbf{c})] \cdot d\mathbf{S} &= \left\{ \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j u_j \right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k c_k \right) \right] \right\} \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l \right) \\
&= \left\{ \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \times \left[\sum_{j=1}^{3} \sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) u_j c_k \right] \right\} \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l \right) \\
&= \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mjk} u_j c_k \right) \right] \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l \right) \\
&= \left[\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{m=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_m) \varepsilon_{mjk} \frac{\partial}{\partial x_i} (u_j c_k) \right] \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l \right) \\
&= \left[\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{m=1}^{3} \sum_{n=1}^{3} \boldsymbol{\delta}_n \varepsilon_{nim} \varepsilon_{mjk} c_k \frac{\partial u_j}{\partial x_i} \right] \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \sum_{n=1}^{3} (\boldsymbol{\delta}_n \cdot \boldsymbol{\delta}_l) \varepsilon_{nim} \varepsilon_{mjk} c_k \frac{\partial u_j}{\partial x_i}\, dS_l\end{aligned}$$

Continue the simplification.

$$
\begin{aligned}
[\nabla \times (\mathbf{u} \times \mathbf{c})] \cdot d\mathbf{S} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} \delta_{nl}\varepsilon_{nim}\varepsilon_{mjk}c_k \frac{\partial u_j}{\partial x_i}\, dS_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{lim}\varepsilon_{mjk}c_k \frac{\partial u_j}{\partial x_i}\, dS_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} \delta_{kn}\varepsilon_{lim}\varepsilon_{mjk}c_n \frac{\partial u_j}{\partial x_i}\, dS_l \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} (\boldsymbol{\delta}_k \cdot \boldsymbol{\delta}_n)\varepsilon_{lim}\varepsilon_{mjk}c_n \frac{\partial u_j}{\partial x_i}\, dS_l \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_k\varepsilon_{lim}\varepsilon_{mjk} \frac{\partial u_j}{\partial x_i}\, dS_l\right) \cdot \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n c_n\right) \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} (\boldsymbol{\delta}_m \times \boldsymbol{\delta}_j)\varepsilon_{lim} \frac{\partial u_j}{\partial x_i}\, dS_l\right] \cdot \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n c_n\right) \\
&= \left[\left(\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m\varepsilon_{lim}\, dS_l\, \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j u_j\right)\right] \cdot \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n c_n\right) \\
&= \left\{\left[\sum_{i=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_l \times \boldsymbol{\delta}_i)\, dS_l\, \frac{\partial}{\partial x_i}\right] \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j u_j\right)\right\} \cdot \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n c_n\right) \\
&= \left\{\left[\left(\sum_{l=1}^{3} \boldsymbol{\delta}_l dS_l\right) \times \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right)\right] \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j u_j\right)\right\} \cdot \left(\sum_{n=1}^{3} \boldsymbol{\delta}_n c_n\right) \\
&= [(d\mathbf{S} \times \nabla) \times \mathbf{u}] \cdot \mathbf{c}
\end{aligned}
$$

As a result, equation (3) becomes

$$\iint_S [(d\mathbf{S} \times \nabla) \times \mathbf{u}] \cdot \mathbf{c} = \oint_{\text{bdy } S} (-\mathbf{u} \times d\mathbf{l}) \cdot \mathbf{c}.$$

Since $\mathbf{c}$ is a constant vector, it can be brought outside the integral.

$$\left[\iint_S (d\mathbf{S} \times \nabla) \times \mathbf{u}\right] \cdot \mathbf{c} = \left(-\oint_{\text{bdy } S} \mathbf{u} \times d\mathbf{l}\right) \cdot \mathbf{c}$$

Therefore,

$$\iint_S (d\mathbf{S} \times \nabla) \times \mathbf{u} = -\oint_{\text{bdy } S} \mathbf{u} \times d\mathbf{l}.$$