Problem 1.8

(a) Prove that the two-dimensional rotation matrix (Eq. 1.29) preserves dot products. (That is, show that $\overline{A}_y\overline{B}_y + \overline{A}_z\overline{B}_z = A_yB_y + A_zB_z$.)

(b) What constraints must the elements (R_{ij}) of the three-dimensional rotation matrix (Eq. 1.30) satisfy, in order to preserve the length of $\mathbf{A}$ (for all vectors $\mathbf{A}$)?

Solution

Part (a)

Suppose there's a vector $\mathbf{A} = \langle A_y, A_z\rangle$ in the yz-plane. If the yz-coordinate system is rotated counterclockwise by an angle ϕ, then the vector components in this new $\bar{y}\bar{z}$-coordinate system are given by

$$\overline{A}_y = A_y\cos\phi + A_z\sin\phi$$

$$\overline{A}_z = -A_y\sin\phi + A_z\cos\phi.$$

Similarly, for a vector $\mathbf{B} = \langle B_y, B_z\rangle$,

$$\overline{B}_y = B_y\cos\phi + B_z\sin\phi$$

$$\overline{B}_z = -B_y\sin\phi + B_z\cos\phi.$$

Therefore,

$$\begin{aligned}
\overline{\mathbf{A}}\cdot\overline{\mathbf{B}} &= \overline{A}_y\overline{B}_y + \overline{A}_z\overline{B}_z \\
&= (A_y\cos\phi + A_z\sin\phi)(B_y\cos\phi + B_z\sin\phi) \\
&\qquad\qquad + (-A_y\sin\phi + A_z\cos\phi)(-B_y\sin\phi + B_z\cos\phi) \\
&= (A_yB_y\cos^2\phi + \cancel{A_yB_z\sin\phi\cos\phi} + \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\sin^2\phi) \\
&\qquad\qquad + (A_yB_y\sin^2\phi - \cancel{A_yB_z\sin\phi\cos\phi} - \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\cos^2\phi) \\
&= A_yB_y(\cos^2\phi + \sin^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi) \\
&= A_yB_y + A_zB_z \\
&= \mathbf{A}\cdot\mathbf{B},
\end{aligned}$$

which means the dot product is preserved under rotations.

Part (b)

Eq. 1.30 gives the components of vector $\mathbf{A} = \langle A_x, A_y, A_z \rangle$ in a rotated $\bar{x}\bar{y}\bar{z}$-coordinate system.

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.30}$$

That is,

$$\begin{cases} \overline{A}_x = R_{xx}A_x + R_{xy}A_y + R_{xz}A_z \\ \overline{A}_y = R_{yx}A_x + R_{yy}A_y + R_{yz}A_z \\ \overline{A}_z = R_{zx}A_x + R_{zy}A_y + R_{zz}A_z \end{cases}.$$

For the length of $\mathbf{A}$ to be preserved, it's necessary that $\sqrt{\overline{A}_x^2 + \overline{A}_y^2 + \overline{A}_z^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$, or

$$\overline{A}_x^2 + \overline{A}_y^2 + \overline{A}_z^2 = A_x^2 + A_y^2 + A_z^2. \tag{1}$$

Evaluate the left side.

$$\begin{aligned} \overline{A}_x^2 + \overline{A}_y^2 + \overline{A}_z^2 &= (R_{xx}A_x + R_{xy}A_y + R_{xz}A_z)^2 \\ &\quad + (R_{yx}A_x + R_{yy}A_y + R_{yz}A_z)^2 \\ &\quad + (R_{zx}A_x + R_{zy}A_y + R_{zz}A_z)^2 \\ &= A_x^2(R_{xx}^2 + R_{yx}^2 + R_{zx}^2) + A_y^2(R_{xy}^2 + R_{yy}^2 + R_{zy}^2) + A_z^2(R_{xz}^2 + R_{yz}^2 + R_{zz}^2) \\ &\quad + 2A_xA_y(R_{xx}R_{xy} + R_{yx}R_{yy} + R_{zx}R_{zy}) \\ &\quad + 2A_xA_z(R_{xx}R_{xz} + R_{yx}R_{yz} + R_{zx}R_{zz}) \\ &\quad + 2A_yA_z(R_{xy}R_{xz} + R_{yy}R_{yz} + R_{zy}R_{zz}) \end{aligned}$$

For equation (1) to be satisfied, then,

$$\left.\begin{aligned} R_{xx}^2 + R_{yx}^2 + R_{zx}^2 &= 1 \\ R_{xy}^2 + R_{yy}^2 + R_{zy}^2 &= 1 \\ R_{xz}^2 + R_{yz}^2 + R_{zz}^2 &= 1 \\ R_{xx}R_{xy} + R_{yx}R_{yy} + R_{zx}R_{zy} &= 0 \\ R_{xx}R_{xz} + R_{yx}R_{yz} + R_{zx}R_{zz} &= 0 \\ R_{xy}R_{xz} + R_{yy}R_{yz} + R_{zy}R_{zz} &= 0 \end{aligned}\right\}.$$