Problem 2.13

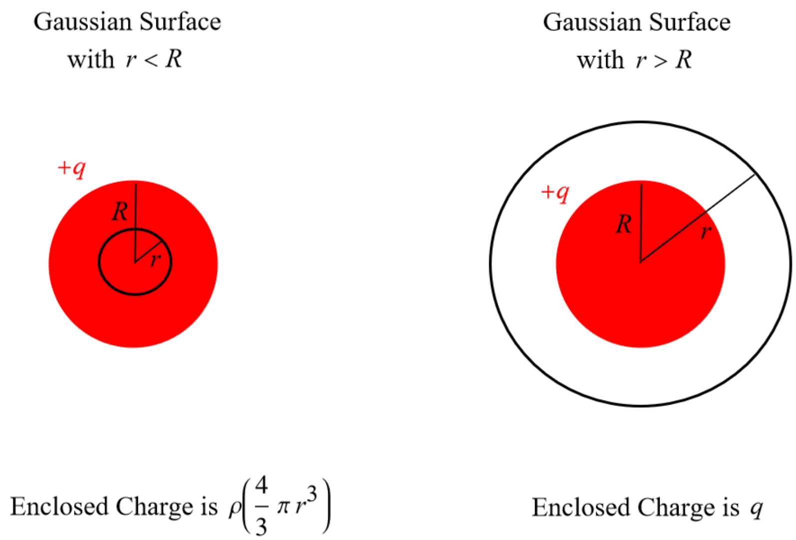
Use Gauss's law to find the electric field inside a uniformly charged solid sphere (charge density ρ). Compare your answer to Prob. 2.8.

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the spherical symmetry, the divergence is sufficient. Integrate both sides over the volume of a (black) concentric spherical Gaussian surface with radius r. Two cases need to be considered: (1) $r < R$ and (2) $r > R$.



$$\iiint\limits_{x_0^2+y_0^2+z_0^2 \leq r^2} \nabla \cdot \mathbf{E}\, dV_0 = \iiint\limits_{x_0^2+y_0^2+z_0^2 \leq r^2} \frac{\rho}{\epsilon_0}\, dV_0$$

$$= \frac{1}{\epsilon_0} \iiint\limits_{x_0^2+y_0^2+z_0^2 \leq r^2} \rho\, dV_0$$

$$= \begin{cases} \dfrac{1}{\epsilon_0}\rho\left(\dfrac{4}{3}\pi r^3\right) & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Apply the divergence theorem on the left side.

$$\oiint_{x_0^2+y_0^2+z_0^2=r^2} \mathbf{E}\cdot d\mathbf{S}_0 = \begin{cases} \dfrac{\rho}{\epsilon_0}\left(\dfrac{4}{3}\pi r^3\right) & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Because of the spherical symmetry, the electric field is expected to be entirely radial: $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\oiint_{r_0^2=r^2} [E(r_0)\hat{\mathbf{r}}_0]\cdot(\hat{\mathbf{r}}_0\, dS_0) = \begin{cases} \left(\dfrac{q}{\frac{4}{3}\pi R^3}\right)\dfrac{1}{\epsilon_0}\left(\dfrac{4}{3}\pi r^3\right) & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Evaluate the dot product.

$$\oiint_{r_0=r} E(r)\, dS_0 = \begin{cases} \dfrac{q}{\epsilon_0}\dfrac{r^3}{R^3} & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

$E(r)$ is constant on the spherical Gaussian surface, so it can be pulled in front of the integral.

$$E(r)\oiint_{r_0=r} dS_0 = \begin{cases} \dfrac{q}{\epsilon_0}\dfrac{r^3}{R^3} & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Evaluate the surface integral.

$$E(r)(4\pi r^2) = \begin{cases} \dfrac{q}{\epsilon_0}\dfrac{r^3}{R^3} & \text{if } r < R \\ \dfrac{q}{\epsilon_0} & \text{if } r > R \end{cases}$$

Solve for $E(r)$.

$$E(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R^3}r & \text{if } r < R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2} & \text{if } r > R \end{cases}$$

Therefore, the electric field around the solid ball is

$$\mathbf{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R^3}r\hat{\mathbf{r}} & \text{if } r < R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$

This is the same result obtained in Problem 2.8 but with r instead of z. In terms of the given charge density $\rho = q/\left(\frac{4}{3}\pi R^3\right)$, it is

$$\mathbf{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{\rho\left(\frac{4}{3}\pi R^3\right)}{R^3} r\hat{\mathbf{r}} & \text{if } r < R \\[2ex] \dfrac{1}{4\pi\epsilon_0}\dfrac{\rho\left(\frac{4}{3}\pi R^3\right)}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases} = \begin{cases} \dfrac{\rho}{3\epsilon_0} r\hat{\mathbf{r}} & \text{if } r < R \\[2ex] \dfrac{\rho}{3\epsilon_0}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$