Problem 2.15

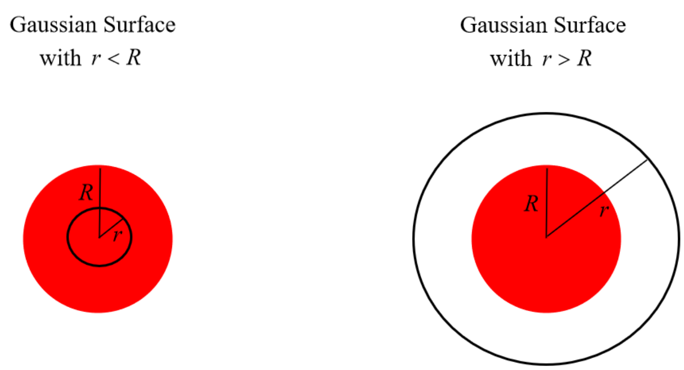
Find the electric field inside a sphere that carries a charge density proportional to the distance from the center, $\rho = kr$, for some constant k. [*Caution:* This charge density is *not* uniform, and you must *integrate* to get the enclosed charge.]

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the spherical symmetry, the divergence is sufficient. Suppose the sphere has radius R and integrate both sides over the volume of a (black) concentric spherical Gaussian surface with radius r. Two cases need to be considered: (1) $r < R$ and (2) $r > R$.



$$\iiint\limits_{x_0^2+y_0^2+z_0^2\leq r^2} \nabla \cdot \mathbf{E}\, dV_0 = \iiint\limits_{x_0^2+y_0^2+z_0^2\leq r^2} \frac{\rho}{\epsilon_0}\, dV_0$$

$$= \frac{1}{\epsilon_0} \iiint\limits_{x_0^2+y_0^2+z_0^2\leq r^2} \rho\, dV_0$$

$$= \begin{cases} \dfrac{1}{\epsilon_0}\displaystyle\int_0^\pi \int_0^{2\pi} \int_0^r kr_0(r_0^2 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0) & \text{if } r < R \\ \dfrac{1}{\epsilon_0}\displaystyle\int_0^\pi \int_0^{2\pi} \int_0^R kr_0(r_0^2 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0) & \text{if } r > R \end{cases}$$

$$= \begin{cases} \dfrac{k}{\epsilon_0}\left(\displaystyle\int_0^\pi \sin\theta_0\, d\theta_0\right)\left(\displaystyle\int_0^{2\pi} d\phi_0\right)\left(\displaystyle\int_0^r r_0^3\, dr_0\right) & \text{if } r < R \\ \dfrac{k}{\epsilon_0}\left(\displaystyle\int_0^\pi \sin\theta_0\, d\theta_0\right)\left(\displaystyle\int_0^{2\pi} d\phi_0\right)\left(\displaystyle\int_0^R r_0^3\, dr_0\right) & \text{if } r > R \end{cases}$$

Apply the divergence theorem on the left side and evaluate the integrals on the right side.

$$\oiint_{x_0^2+y_0^2+z_0^2=r^2} \mathbf{E}\cdot d\mathbf{S}_0 = \begin{cases} \dfrac{k}{\epsilon_0}(2)(2\pi)\left(\dfrac{r^4}{4}\right) & \text{if } r < R \\ \dfrac{k}{\epsilon_0}(2)(2\pi)\left(\dfrac{R^4}{4}\right) & \text{if } r > R \end{cases}$$

Because of the spherical symmetry, the electric field is expected to be entirely radial: $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\oiint_{r_0^2=r^2} [E(r_0)\hat{\mathbf{r}}_0]\cdot(\hat{\mathbf{r}}_0\, dS_0) = \begin{cases} \dfrac{\pi k}{\epsilon_0}r^4 & \text{if } r < R \\ \dfrac{\pi k}{\epsilon_0}R^4 & \text{if } r > R \end{cases}$$

Evaluate the dot product.

$$\oiint_{r_0=r} E(r)\, dS_0 = \begin{cases} \dfrac{\pi k}{\epsilon_0}r^4 & \text{if } r < R \\ \dfrac{\pi k}{\epsilon_0}R^4 & \text{if } r > R \end{cases}$$

$E(r)$ is constant on the spherical Gaussian surface, so it can be pulled in front of the integral.

$$E(r)\oiint_{r_0=r} dS_0 = \begin{cases} \dfrac{\pi k}{\epsilon_0}r^4 & \text{if } r < R \\ \dfrac{\pi k}{\epsilon_0}R^4 & \text{if } r > R \end{cases}$$

Evaluate the surface integral.

$$E(r)(4\pi r^2) = \begin{cases} \dfrac{\pi k}{\epsilon_0}r^4 & \text{if } r < R \\ \dfrac{\pi k}{\epsilon_0}R^4 & \text{if } r > R \end{cases}$$

Solve for $E(r)$.

$$E(r) = \begin{cases} \dfrac{k}{4\epsilon_0}r^2 & \text{if } r < R \\ \dfrac{k}{4\epsilon_0}\dfrac{R^4}{r^2} & \text{if } r > R \end{cases}$$

Therefore, the electric field around the solid ball with charge density $\rho = kr$ is

$$\mathbf{E} = \begin{cases} \dfrac{k}{4\epsilon_0}r^2\hat{\mathbf{r}} & \text{if } r < R \\ \dfrac{k}{4\epsilon_0}\dfrac{R^4}{r^2}\hat{\mathbf{r}} & \text{if } r > R \end{cases}.$$