Problem 2.20

Calculate $\nabla \times \mathbf{E}$ directly from Eq. 2.8, by the method of Section 2.2.2. Refer to Prob. 1.63b if you get stuck.

Solution

Eq. 2.8 on page 63 gives the electric field for a continuous distribution of charge in space.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau' \tag{2.8}$$

Take the curl of both sides and bring the constant in front.

$$\begin{aligned}\nabla \times \mathbf{E} &= \nabla \times \left[\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'\right]\\ &= \frac{1}{4\pi\epsilon_0}\left[\nabla \times \int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'\right]\end{aligned}$$

The curl is taken with respect to the unprimed variables (x, y, z), so it can be brought inside the integral, which is taken over the primed variables (x', y', z').

$$\nabla \times \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla \times \left[\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\right] d\tau'$$

$\rho(\mathbf{r}')$ is a function of the primed variables, so it can be brought outside the curl.

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\nabla \times \left(\frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\right)\right] d\tau'\\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\nabla \times \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)\right] d\tau'\\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\nabla \times \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau'\\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\nabla \times \left(\frac{\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau'\\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\left(\nabla \times \frac{\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}\right) - \left(\nabla \times \frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau'\end{aligned}$$

In each set of parentheses is the curl of a scalar function times a vector function, so use Identity (7).

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left\{\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}(\nabla \times \mathbf{r}) - \mathbf{r} \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right]\right.\\ &\qquad \left. - \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\underbrace{(\nabla \times \mathbf{r}')}_{=\,\mathbf{0}} - \mathbf{r}' \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right]\right\} d\tau'\\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}(\nabla \times \mathbf{r}) - (\mathbf{r}-\mathbf{r}') \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau' \qquad (1)\end{aligned}$$

From Problem 1.63,

$$\nabla \times (r^n \hat{\mathbf{r}}) = \mathbf{0}.$$

Set $n = 1$.

$$\nabla \times (r\hat{\mathbf{r}}) = \mathbf{0}$$

$$\nabla \times \left[r\left(\frac{\mathbf{r}}{r}\right)\right] = \mathbf{0}$$

$$\nabla \times \mathbf{r} = \mathbf{0}$$

Consequently, equation (1) becomes

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}(\overbrace{\nabla \times \mathbf{r}}^{=\mathbf{0}}) - (\mathbf{r}-\mathbf{r}') \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau' \\ &= -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[(\mathbf{r}-\mathbf{r}') \times \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right] d\tau'. \qquad (2)\end{aligned}$$

Calculate the gradient in parentheses.

$$\begin{aligned}\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|^3} &= \left\langle \frac{\partial}{\partial x}\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}, \frac{\partial}{\partial y}\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}, \frac{\partial}{\partial z}\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right\rangle \\ &= \left\langle \frac{-3(x-x')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}}, \frac{-3(y-y')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}}, \frac{-3(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}}\right\rangle \\ &= -\frac{3}{[(x-x')^2+(y-y')^2+(z-z')^2]^{5/2}}\langle x-x', y-y', z-z'\rangle \\ &= -\frac{3}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}')\end{aligned}$$

Therefore, equation (2) becomes

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[(\mathbf{r}-\mathbf{r}') \times \left(-\frac{3}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}')\right)\right] d\tau' \\ &= \frac{3}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5}[\underbrace{(\mathbf{r}-\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}_{=\mathbf{0}}]\, d\tau' = \mathbf{0}.\end{aligned}$$