Problem 2.24

For the charge configuration of Prob. 2.16, find the potential at the center, using infinity as your reference point.

Solution

An electrostatic field must satisfy $\nabla \times \mathbf{E} = \mathbf{0}$, which implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). To solve for V, integrate both sides along a path between two points in space with position vectors, $\mathbf{a}$ and $\mathbf{b}$, and use the fundamental theorem for gradients.

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}_0 &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{b}) - V(\mathbf{a})] \\ &= V(\mathbf{a}) - V(\mathbf{b})\end{aligned}$$

In this context $\mathbf{a}$ is the position vector for the reference point (taken to be infinity $\boldsymbol{\infty}$ here), and $\mathbf{b}$ is the position vector $\mathbf{r}$ for the point we're interested in knowing the electric potential.

$$\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}_0 = V(\boldsymbol{\infty}) - V(\mathbf{r})$$

The potential at the reference point is taken to be zero: $V(\boldsymbol{\infty}) = 0$.

$$\int_{\boldsymbol{\infty}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}_0 = -V(\mathbf{r})$$

Therefore, the potential at $\mathbf{r} = \langle x, y, z \rangle$ is

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\boldsymbol{\infty}} \mathbf{E} \cdot d\mathbf{l}_0.$$

According to Problem 2.16, the electric field around a thick spherical shell with charge density $\rho = k/r^2$ for $a \le r \le b$ is

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } r < a \\ \dfrac{k}{\epsilon_0}\left(\dfrac{r-a}{r^2}\right)\hat{\mathbf{r}} & \text{if } a < r < b \\ \dfrac{k}{\epsilon_0}\left(\dfrac{b-a}{r^2}\right)\hat{\mathbf{r}} & \text{if } r > b \end{cases}.$$

Since the electric field is spherically symmetric, the path taken from $\mathbf{r}$ to $\boldsymbol{\infty}$ is a radial one and parameterized by r_0, where $r \le r_0 < \infty$.

$$V(r) = \int_r^\infty \mathbf{E}(r_0) \cdot d\mathbf{r}_0$$

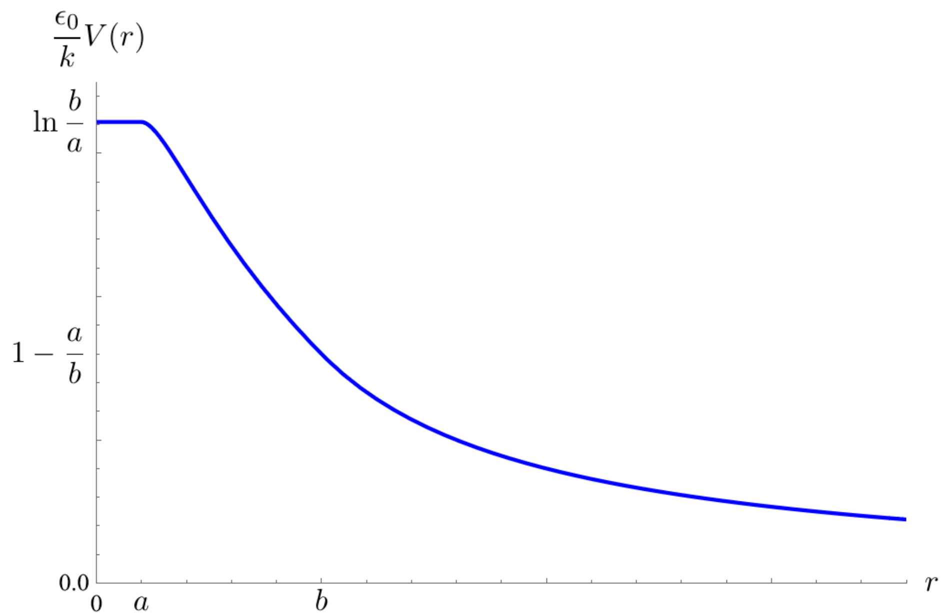
Evaluate the dot product, substitute the formula for the electric field, evaluate the integrals, and simplify the result.

$$
\begin{aligned}
V(r) &= \int_r^\infty [E(r_0)\hat{\mathbf{r}}_0]\cdot(\hat{\mathbf{r}}_0\,dr_0) \\
&= \int_r^\infty E(r_0)\,dr_0 \\
&= \begin{cases} \displaystyle\int_r^a (0)\,dr_0 + \int_a^b \frac{k}{\epsilon_0}\left(\frac{r_0-a}{r_0^2}\right)dr_0 + \int_b^\infty \frac{k}{\epsilon_0}\left(\frac{b-a}{r_0^2}\right)dr_0 & \text{if } r<a \\ \displaystyle\int_r^b \frac{k}{\epsilon_0}\left(\frac{r_0-a}{r_0^2}\right)dr_0 + \int_b^\infty \frac{k}{\epsilon_0}\left(\frac{b-a}{r_0^2}\right)dr_0 & \text{if } a<r<b \\ \displaystyle\int_r^\infty \frac{k}{\epsilon_0}\left(\frac{b-a}{r_0^2}\right)dr_0 & \text{if } r>b \end{cases} \\
&= \begin{cases} \displaystyle\frac{k}{\epsilon_0}\left(\int_a^b \frac{dr_0}{r_0} - a\int_a^b \frac{dr_0}{r_0^2}\right) + \frac{k}{\epsilon_0}(b-a)\int_b^\infty \frac{dr_0}{r_0^2} & \text{if } r<a \\ \displaystyle\frac{k}{\epsilon_0}\left(\int_r^b \frac{dr_0}{r_0} - a\int_r^b \frac{dr_0}{r_0^2}\right) + \frac{k}{\epsilon_0}(b-a)\int_b^\infty \frac{dr_0}{r_0^2} & \text{if } a<r<b \\ \displaystyle\frac{k}{\epsilon_0}(b-a)\int_r^\infty \frac{dr_0}{r_0^2} & \text{if } r>b \end{cases} \\
&= \begin{cases} \displaystyle\frac{k}{\epsilon_0}\left[\ln r_0\Big|_a^b - a\left(-\frac{1}{r_0}\right)\Big|_a^b\right] + \frac{k}{\epsilon_0}(b-a)\left(-\frac{1}{r_0}\right)\Big|_b^\infty & \text{if } r<a \\ \displaystyle\frac{k}{\epsilon_0}\left[\ln r_0\Big|_r^b - a\left(-\frac{1}{r_0}\right)\Big|_r^b\right] + \frac{k}{\epsilon_0}(b-a)\left(-\frac{1}{r_0}\right)\Big|_b^\infty & \text{if } a<r<b \\ \displaystyle\frac{k}{\epsilon_0}(b-a)\left(-\frac{1}{r_0}\right)\Big|_r^\infty & \text{if } r>b \end{cases} \\
&= \begin{cases} \displaystyle\frac{k}{\epsilon_0}\left[(\ln b - \ln a) - a\left(-\frac{1}{b}+\frac{1}{a}\right)\right] + \frac{k}{\epsilon_0}(b-a)\left(\frac{1}{b}\right) & \text{if } r<a \\ \displaystyle\frac{k}{\epsilon_0}\left[(\ln b - \ln r) - a\left(-\frac{1}{b}+\frac{1}{r}\right)\right] + \frac{k}{\epsilon_0}(b-a)\left(\frac{1}{b}\right) & \text{if } a<r<b \\ \displaystyle\frac{k}{\epsilon_0}(b-a)\left(\frac{1}{r}\right) & \text{if } r>b \end{cases}
\end{aligned}
$$

Therefore,

$$V(r) = \begin{cases} \dfrac{k}{\epsilon_0}\left(\ln\dfrac{b}{a}\right) & \text{if } r < a \\[2ex] \dfrac{k}{\epsilon_0}\left(\ln\dfrac{b}{r} + 1 - \dfrac{a}{r}\right) & \text{if } a < r < b \\[2ex] \dfrac{k}{\epsilon_0}\left(\dfrac{b-a}{r}\right) & \text{if } r > b \end{cases}.$$

Below is a plot of $(\epsilon_0/k)V(r)$ versus r.



The electric potential at the center of the spherical shell is

$$V(0) = \frac{k}{\epsilon_0}\left(\ln\frac{b}{a}\right).$$