Problem 2.26

Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$: What then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

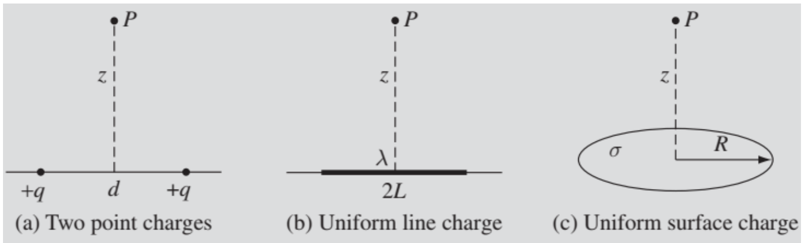


Fig. 2.34

Solution

The two equations governing the electric (electrostatic) field are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{2}$$

Equation (2) implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). Plugging this into equation (1) yields Poisson's equation for the electric potential.

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\epsilon_0}$$

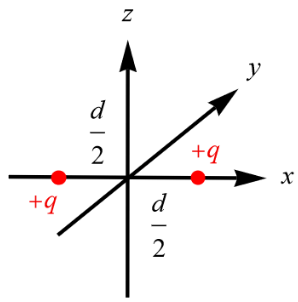
$$-\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

All of electrostatics comes down to solving this linear inhomogeneous PDE.

Part (a)

For the two point charges illustrated in part (a), the charge density can be represented using Dirac delta functions.



Poisson's equation becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\left[q\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z) + q\delta\left(x-\frac{d}{2}\right)\delta(y)\delta(z)\right], \quad -\infty < x, y, z < \infty$$

$$= -\frac{q}{\epsilon_0}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x-\frac{d}{2}\right)\delta(y)\delta(z).$$

Make the substitution $V = U + W$.

$$\nabla^2(U+W) = -\frac{q}{\epsilon_0}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x-\frac{d}{2}\right)\delta(y)\delta(z)$$

$$\nabla^2 U + \nabla^2 W = -\frac{q}{\epsilon_0}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z) - \frac{q}{\epsilon_0}\delta\left(x-\frac{d}{2}\right)\delta(y)\delta(z)$$

If we set

$$\nabla^2 U = -\frac{q}{\epsilon_0}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z), \quad -\infty < x, y, z < \infty \tag{3}$$

then the previous equation becomes

$$\nabla^2 W = -\frac{q}{\epsilon_0}\delta\left(x-\frac{d}{2}\right)\delta(y)\delta(z), \quad -\infty < x, y, z < \infty. \tag{4}$$

Substituting $V = U + W$ is essentially invoking the principle of superposition: The potential at $\mathbf{r}$ in the two-charge system is the sum of the potentials from each charge individually. The reason it works is because

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is a linear operator. Away from $\left(-\frac{d}{2}, 0, 0\right)$ the right side of equation (3) is zero.

$$\nabla^2 U = 0, \quad (x, y, z) \neq \left(-\frac{d}{2}, 0, 0\right)$$

The potential around the charge at $\left(-\frac{d}{2}, 0, 0\right)$ is spherically symmetric with respect to this point: $U = U(\mathscr{r})$, where $\mathscr{r} = \sqrt{(x+d/2)^2 + (y-0)^2 + (z-0)^2}$ is the radial distance from $\left(-\frac{d}{2}, 0, 0\right)$.

Expand ∇^2 in spherical coordinates.

$$\frac{1}{\mathscr{r}^2}\frac{d}{d\mathscr{r}}\left(\mathscr{r}^2\frac{dU}{d\mathscr{r}}\right)=0$$

Multiply both sides by $\mathscr{r}^2$.

$$\frac{d}{d\mathscr{r}}\left(\mathscr{r}^2\frac{dU}{d\mathscr{r}}\right)=0$$

Integrate both sides with respect to $\mathscr{r}$.

$$\mathscr{r}^2\frac{dU}{d\mathscr{r}}=C_1$$

Divide both sides by $\mathscr{r}^2$.

$$\frac{dU}{d\mathscr{r}}=\frac{C_1}{\mathscr{r}^2}$$

Integrate both sides with respect to $\mathscr{r}$ again.

$$U(\mathscr{r})=-\frac{C_1}{\mathscr{r}}+C_2$$

To determine C_1, integrate both sides of equation (3) over the volume of a sphere centered at $\left(-\frac{d}{2},0,0\right)$ with radius ε.

$$\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2}\nabla^2U\,d\tau=\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2}\frac{-q}{\epsilon_0}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z)\,d\tau$$

$$\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2}\nabla\cdot\nabla U\,d\tau=-\frac{q}{\epsilon_0}\underbrace{\iiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2\leq\varepsilon^2}\delta\left(x+\frac{d}{2}\right)\delta(y)\delta(z)\,d\tau}_{=1}$$

Apply the divergence theorem on the left.

$$\oiint\limits_{\left(x+\frac{d}{2}\right)^2+y^2+z^2=\varepsilon^2}\nabla U\cdot d\mathbf{S}=-\frac{q}{\epsilon_0}$$

Switch to spherical coordinates $(\mathscr{r},\phi,\theta)$, where θ is the angle from the polar axis.

$$\int_0^{\pi}\int_0^{2\pi}\left[\frac{dU}{d\mathscr{r}}(\varepsilon)\hat{\boldsymbol{\mathscr{r}}}\right]\cdot\left(\hat{\boldsymbol{\mathscr{r}}}\varepsilon^2\sin\theta\,d\phi\,d\theta\right)=-\frac{q}{\epsilon_0}$$

$$\varepsilon^2\frac{dU}{d\mathscr{r}}(\varepsilon)\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\phi\,d\theta=-\frac{q}{\epsilon_0}$$

$$\varepsilon^2\frac{dU}{d\mathscr{r}}(\varepsilon)\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right)=-\frac{q}{\epsilon_0}$$

$$\varepsilon^2\left(\frac{C_1}{\varepsilon^2}\right)(2)(2\pi)=-\frac{q}{\epsilon_0}$$

Solve for C_1.

$$C_1 = -\frac{q}{4\pi\epsilon_0}$$

Consequently, the potential due to the charge at $\left(-\frac{d}{2}, 0, 0\right)$ alone is

$$U(\mathscr{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} + C_2,$$

where C_2 is the arbitrary additive constant of electric potential. It's chosen to be zero as $\mathscr{r} \to \infty$.

$$\lim_{\mathscr{r}\to\infty} U(\mathscr{r}) = 0 \quad \to \quad C_2 = 0$$

As a result,

$$U(\mathscr{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}},$$

or in terms of the original variables,

$$U(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{\left(x + \frac{d}{2}\right)^2 + y^2 + z^2}}.$$

Using the same argument for the charge $+q$ at $\left(\frac{d}{2}, 0, 0\right)$,

$$W(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{\left(x - \frac{d}{2}\right)^2 + y^2 + z^2}}.$$

Therefore, the electric potential of the two-charge system is

$$\begin{aligned} V(x, y, z) &= U(x, y, z) + W(x, y, z) \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\left(x + \frac{d}{2}\right)^2 + y^2 + z^2}} + \frac{1}{\sqrt{\left(x - \frac{d}{2}\right)^2 + y^2 + z^2}}\right], \end{aligned}$$

and the electric field is

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle \\ &= \frac{q}{4\pi\epsilon_0}\left\langle \frac{x + \frac{d}{2}}{\left[\left(x + \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}} + \frac{x - \frac{d}{2}}{\left[\left(x - \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}}, \right. \\ &\qquad \frac{y}{\left[\left(x + \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}} + \frac{y}{\left[\left(x - \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}}, \\ &\qquad \left. \frac{z}{\left[\left(x + \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}} + \frac{z}{\left[\left(x - \frac{d}{2}\right)^2 + y^2 + z^2\right]^{3/2}} \right\rangle. \end{aligned}$$

Set $x = 0$ and $y = 0$ to find the electric potential at point P.

$$V(0,0,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\frac{d^2}{4}+z^2}} + \frac{1}{\sqrt{\frac{d^2}{4}+z^2}}\right) = \frac{q}{4\pi\epsilon_0}\frac{2}{\sqrt{\frac{d^2}{4}+z^2}}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{4}{\sqrt{d^2+4z^2}}$$

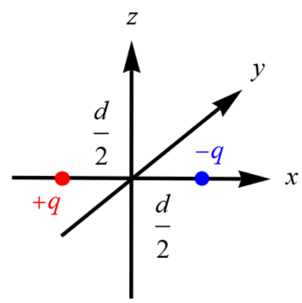
$$= \frac{q}{\pi\epsilon_0}\frac{1}{\sqrt{d^2+4z^2}}$$

And set $x = 0$ and $y = 0$ to find the electric field at point P.

$$\mathbf{E}(0,0,z) = \frac{q}{4\pi\epsilon_0}\left\langle \frac{\frac{d}{2}}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} + \frac{-\frac{d}{2}}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}, \frac{0}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} + \frac{0}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}, \frac{z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} + \frac{z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\right\rangle$$

$$= \frac{q}{4\pi\epsilon_0}\left\langle 0, 0, \frac{2z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\right\rangle$$

This is the result of Example 2.1 on page 62.



If the charge at $\left(\frac{d}{2},0,0\right)$ is $-q$ instead, then

$$W(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{-q}{\sqrt{\left(x-\frac{d}{2}\right)^2+y^2+z^2}},$$

and

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\left(x+\frac{d}{2}\right)^2+y^2+z^2}} - \frac{1}{\sqrt{\left(x-\frac{d}{2}\right)^2+y^2+z^2}}\right],$$

which means

$$\begin{aligned}
\mathbf{E} &= -\left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle \\
&= \frac{q}{4\pi\epsilon_0} \left\langle \frac{x+\frac{d}{2}}{\left[\left(x+\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} - \frac{x-\frac{d}{2}}{\left[\left(x-\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}}, \right. \\
&\qquad \frac{y}{\left[\left(x+\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} - \frac{y}{\left[\left(x-\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}}, \\
&\qquad \left. \frac{z}{\left[\left(x+\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} - \frac{z}{\left[\left(x-\frac{d}{2}\right)^2+y^2+z^2\right]^{3/2}} \right\rangle.
\end{aligned}$$

Notice that the electric potential is zero at $x = 0$ on the entire yz-plane, not just at P.

$$V(0, y, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\frac{d^2}{4}+y^2+z^2}} - \frac{1}{\sqrt{\frac{d^2}{4}+y^2+z^2}}\right] = 0$$

Remember this—it's very important because boundary value problems involving Poisson's equation can be solved with a grounded plane by considering two opposite charges equidistant from this plane on opposite sides. Set $x = 0$ and $y = 0$ to find the electric potential at point P.

$$\begin{aligned}
\mathbf{E}(0,0,z) &= \frac{q}{4\pi\epsilon_0} \left\langle \frac{\frac{d}{2}}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} - \frac{-\frac{d}{2}}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}, \right. \\
&\qquad \frac{0}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} - \frac{0}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}, \\
&\qquad \left. \frac{z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} - \frac{z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}} \right\rangle. \\
&= \frac{q}{4\pi\epsilon_0}\left\langle \frac{d}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}, 0, 0 \right\rangle
\end{aligned}$$

This is the result of Problem 2.2. There are no discrepancies.

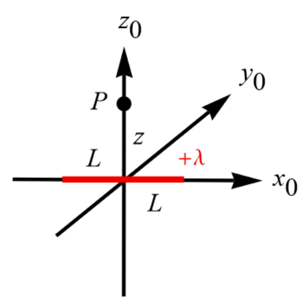
Part (b)

From part (a), the electric potential at a radial distance $\mathscr{r}$ from a point charge q is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}.$$

Use it to obtain the formulas for a continuous distribution of charge.

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}} \quad \Rightarrow \quad V(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl' & \text{for a line charge density} \\ \frac{1}{4\pi\epsilon_0}\iint \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' & \text{for a surface charge density} \\ \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' & \text{for a volume charge density} \end{cases}$$



Note that $\mathbf{r}' = \langle x', 0, 0\rangle$ is the position vector to a point on the charged rod, and $\mathbf{r} = \langle x, y, z\rangle$ is the position vector to the point we want to know the electric potential.

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl' \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda}{|\mathbf{r}-\mathbf{r}'|}\,dl' \\ &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda}{|\langle x,y,z\rangle - \langle x',0,0\rangle|}\,dx' \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L} \frac{1}{\sqrt{(x-x')^2+y^2+z^2}}\,dx' \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L} \frac{1}{\sqrt{(x'-x)^2+y^2+z^2}}\,dx' \end{aligned}$$

Make the following substitution.

$$x' - x = \sqrt{y^2+z^2}\tan\theta \quad \rightarrow \quad (x'-x)^2 + y^2 + z^2 = (y^2+z^2)(\tan^2\theta + 1) = (y^2+z^2)\sec^2\theta$$

$$dx' = \sqrt{y^2+z^2}\sec^2\theta\,d\theta$$

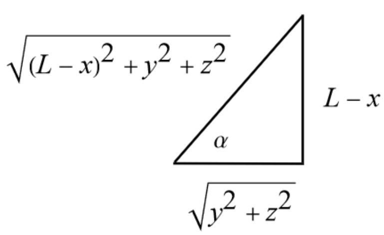
Consequently,

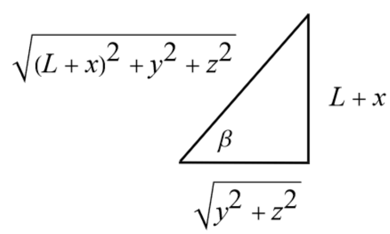
$$\begin{aligned}
V(x,y,z) &= \frac{\lambda}{4\pi\epsilon_0}\int_{\tan^{-1}\left(\frac{-L-x}{\sqrt{y^2+z^2}}\right)}^{\tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right)} \frac{1}{\sqrt{y^2+z^2}\sec\theta}\left(\sqrt{y^2+z^2}\sec^2\theta\,d\theta\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_{-\tan^{-1}\left(\frac{L+x}{\sqrt{y^2+z^2}}\right)}^{\tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right)} \sec\theta\,d\theta \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln|\sec\theta+\tan\theta|\Big|_{-\tan^{-1}\left(\frac{L+x}{\sqrt{y^2+z^2}}\right)}^{\tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right)} \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left|\sec\tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right)+\tan\tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right)\right|\right. \\
&\qquad\qquad \left.-\ln\left|\sec\tan^{-1}\left(\frac{L+x}{\sqrt{y^2+z^2}}\right)-\tan\tan^{-1}\left(\frac{L+x}{\sqrt{y^2+z^2}}\right)\right|\right].
\end{aligned}$$

Let

$$\alpha = \tan^{-1}\left(\frac{L-x}{\sqrt{y^2+z^2}}\right) \quad \text{and} \quad \beta = \tan^{-1}\left(\frac{L+x}{\sqrt{y^2+z^2}}\right),$$

and draw the implied right triangles to determine the secant and tangent of α and β.





So then

$$\sec\alpha = \frac{\sqrt{(L-x)^2+y^2+z^2}}{\sqrt{y^2+z^2}} \qquad \sec\beta = \frac{\sqrt{(L+x)^2+y^2+z^2}}{\sqrt{y^2+z^2}}$$

$$\tan\alpha = \frac{L-x}{\sqrt{y^2+z^2}} \qquad \tan\beta = \frac{L+x}{\sqrt{y^2+z^2}}$$

and the potential becomes

$$\begin{aligned}
V(x,y,z) &= \frac{\lambda}{4\pi\epsilon_0}\left(\ln|\sec\alpha+\tan\alpha| - \ln|\sec\beta-\tan\beta|\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left|\frac{\sqrt{(L-x)^2+y^2+z^2}}{\sqrt{y^2+z^2}}+\frac{L-x}{\sqrt{y^2+z^2}}\right| - \ln\left|\frac{\sqrt{(L+x)^2+y^2+z^2}}{\sqrt{y^2+z^2}}-\frac{L+x}{\sqrt{y^2+z^2}}\right|\right].
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
V(x,y,z) &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left|\frac{\sqrt{(L-x)^2+y^2+z^2}+L-x}{\sqrt{y^2+z^2}}\right| - \ln\left|\frac{\sqrt{(L+x)^2+y^2+z^2}-L-x}{\sqrt{y^2+z^2}}\right|\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left|\frac{\sqrt{(L-x)^2+y^2+z^2}+L-x}{\sqrt{y^2+z^2}} \times \frac{\sqrt{y^2+z^2}}{\sqrt{(L+x)^2+y^2+z^2}-L-x}\right|\right] \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left|\frac{\sqrt{(L-x)^2+y^2+z^2}+L-x}{\sqrt{(L+x)^2+y^2+z^2}-L-x}\right|
\end{aligned}$$

Therefore, the electric potential is

$$V(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\sqrt{(L-x)^2+y^2+z^2}+L-x}{\sqrt{(L+x)^2+y^2+z^2}-L-x},$$

and the electric field is

$$\begin{aligned}
\mathbf{E} &= -\nabla V \\
&= -\left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle \frac{1}{\sqrt{(L-x)^2+y^2+z^2}} - \frac{1}{\sqrt{(L+x)^2+y^2+z^2}},\right. \\
&\qquad\qquad \frac{y}{y^2+z^2}\left[\frac{L-x}{\sqrt{(L-x)^2+y^2+z^2}} + \frac{L+x}{\sqrt{(L+x)^2+y^2+z^2}}\right], \\
&\qquad\qquad\qquad \left.\frac{z}{y^2+z^2}\left[\frac{L-x}{\sqrt{(L-x)^2+y^2+z^2}} + \frac{L+x}{\sqrt{(L+x)^2+y^2+z^2}}\right]\right\rangle.
\end{aligned}$$

Set $x = 0$ and $y = 0$ to find the electric potential at point P.

$$V(0,0,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\sqrt{L^2+z^2}+L}{\sqrt{L^2+z^2}-L}$$

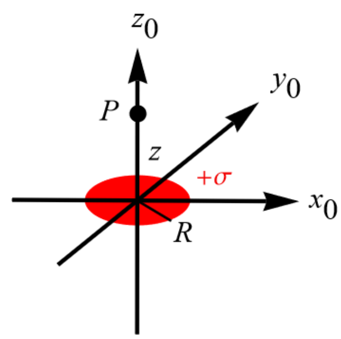
And set $x = 0$ and $y = 0$ to find the electric field at point P.

$$\begin{aligned}
\mathbf{E}(0,0,z) &= \frac{\lambda}{4\pi\epsilon_0}\left\langle \frac{1}{\sqrt{L^2+z^2}} - \frac{1}{\sqrt{L^2+z^2}}, \frac{0}{z^2}\left[\frac{L}{\sqrt{L^2+z^2}} + \frac{L}{\sqrt{L^2+z^2}}\right], \frac{z}{z^2}\left[\frac{L}{\sqrt{L^2+z^2}} + \frac{L}{\sqrt{L^2+z^2}}\right]\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle 0, 0, \frac{2L}{z}\frac{1}{\sqrt{L^2+z^2}}\right\rangle
\end{aligned}$$

This is the result of Example 2.2 on page 64.

Part (c)

Consider now a disk with charge density σ and radius R centered at the origin.



Use the formula in part (b) for a surface charge distribution to calculate the electric potential. Note that $\mathbf{r}' = \langle x', y', 0\rangle$ is the position vector to a point on the charged disk, and $\mathbf{r} = \langle x, y, z\rangle$ is the position vector to the point we want to know the potential.

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\epsilon_0}\iint \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' \\ &= \frac{1}{4\pi\epsilon_0}\iint \frac{\sigma}{|\mathbf{r}-\mathbf{r}'|}\,da' \\ &= \frac{1}{4\pi\epsilon_0}\iint\limits_{x'^2+y'^2\le R^2} \frac{\sigma}{|\langle x,y,z\rangle - \langle x',y',0\rangle|}\,dx'\,dy' \\ &= \frac{\sigma}{4\pi\epsilon_0}\iint\limits_{x'^2+y'^2\le R^2} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\,dx'\,dy' \\ &= \frac{\sigma}{4\pi\epsilon_0}\iint\limits_{x'^2+y'^2\le R^2} \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+z^2}}\,dx'\,dy' \end{aligned}$$

Switch to polar coordinates.

$$x' = r'\cos\theta'$$

$$y' = r'\sin\theta'$$

Consequently,

$$\begin{aligned} V(x,y,z) &= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{(r'\cos\theta'-x)^2+(r'\sin\theta'-y)^2+z^2}}\,(r'\,dr'\,d\theta') \\ &= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\left[\int_0^R \frac{r'}{\sqrt{r'^2-2(x\cos\theta'+y\sin\theta')r'+x^2+y^2+z^2}}\,dr'\right]d\theta'. \end{aligned}$$

We just want the potential at point P, so set $x = 0$ and $y = 0$.

$$\begin{aligned} V(0,0,z) &= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\left(\int_0^R \frac{r'}{\sqrt{r'^2+z^2}}\,dr'\right)d\theta' \\ &= \frac{\sigma}{4\pi\epsilon_0}\left(\int_0^{2\pi} d\theta'\right)\left(\int_0^R \frac{r'}{\sqrt{r'^2+z^2}}\,dr'\right) \end{aligned}$$

Make the following substitution in the second integral.

$$u' = r'^2 + z^2$$

$$du' = 2r'\,dr' \quad \rightarrow \quad \frac{du'}{2} = r'\,dr'$$

As a result, the electric potential at point P is

$$\begin{aligned} V(0,0,z) &= \frac{\sigma}{4\pi\epsilon_0}(2\pi)\left(\int_{z^2}^{R^2+z^2} \frac{1}{\sqrt{u'}}\frac{du'}{2}\right) \\ &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{u'}\Big|_{z^2}^{R^2+z^2}\right) \\ &= \frac{\sigma}{2\epsilon_0}(\sqrt{R^2+z^2} - z). \end{aligned}$$

We can only use $\mathbf{E} = -\nabla V$ to get the z-component of the electric field because only $V(0,0,z)$ is known. The x- and y-components are zero because of symmetry about the z-axis. Therefore, the electric field at point P is

$$\begin{aligned} \mathbf{E}(0,0,z) &= -\left\langle 0, 0, \frac{d}{dz}\frac{\sigma}{2\epsilon_0}(\sqrt{R^2+z^2} - z)\right\rangle \\ &= -\left\langle 0, 0, \frac{\sigma}{2\epsilon_0}\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)\right\rangle \\ &= \left\langle 0, 0, \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\right\rangle, \end{aligned}$$

which is the result of Problem 2.6.