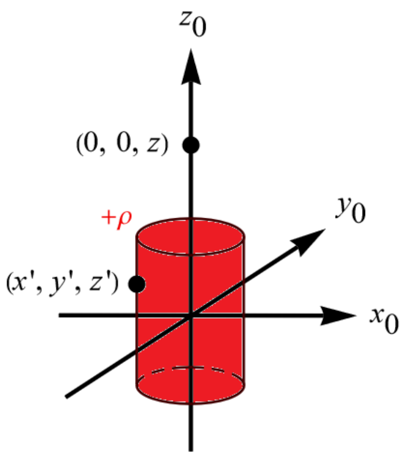
Problem 2.28

Find the potential on the axis of a uniformly charged solid cylinder, a distance z from the center. The length of the cylinder is L, its radius is R, and the charge density is ρ. Use your result to calculate the electric field at this point. (Assume that $z > L/2$.)

Solution

Draw a schematic of the charged cylinder, choosing the origin to be at the center.



$(0,0,z)$ is where we want to know the electric potential, and (x',y',z') is a point on the charged body. Begin with the basic formula for the electric potential of a continuous volume charge distribution.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho}{|\mathbf{r}-\mathbf{r}'|}\, d\tau'$$

For the point $(0,0,z)$ in particular,

$$V(0,0,z) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho}{|\langle 0,0,z\rangle - \langle x',y',z'\rangle|}\, d\tau'$$

$$= \frac{\rho}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{1}{|\langle -x',-y',z-z'\rangle|}\, d\tau'$$

$$= \frac{\rho}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{1}{\sqrt{x'^2+y'^2+(z-z')^2}}\, d\tau'.$$

Since $\mathcal{V}$ is the cylinder here, switch to cylindrical coordinates.

$$x' = r'\cos\phi'$$

$$y' = r'\sin\phi'$$

$$z' = z'$$

Consequently,

$$\begin{aligned} V(0,0,z) &= \frac{\rho}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{(r'\cos\phi')^2+(r'\sin\phi')^2+(z-z')^2}}\,(r'\,dr'\,d\phi'\,dz') \\ &= \frac{\rho}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\left[\int_0^R \frac{r'}{\sqrt{r'^2+(z-z')^2}}\,dr'\right]d\phi'\,dz'. \end{aligned}$$

Make the following substitution.

$$u = r'^2 + (z-z')^2$$

$$du = 2r'\,dr' \quad \rightarrow \quad \frac{du}{2} = r'\,dr'$$

As a result, (note that $z > z'$ because $-L/2 \le z' \le L/2$ and $z > L/2$)

$$\begin{aligned} V(0,0,z) &= \frac{\rho}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\left[\int_{0^2+(z-z')^2}^{R^2+(z-z')^2}\frac{1}{\sqrt{u}}\left(\frac{du}{2}\right)\right]d\phi'\,dz' \\ &= \frac{\rho}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\left[\sqrt{u}\,\Big|_{(z-z')^2}^{R^2+(z-z')^2}\right]d\phi'\,dz' \\ &= \frac{\rho}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_0^{2\pi}\left[\sqrt{R^2+(z-z')^2}-\sqrt{(z-z')^2}\right]d\phi'\,dz' \\ &= \frac{\rho}{4\pi\epsilon_0}\left(\int_0^{2\pi}d\phi'\right)\int_{-L/2}^{L/2}\left[\sqrt{R^2+(z-z')^2}-|z-z'|\right]dz' \\ &= \frac{\rho}{4\pi\epsilon_0}(2\pi)\int_{-L/2}^{L/2}\left[\sqrt{R^2+(z-z')^2}-(z-z')\right]dz' \\ &= \frac{\rho}{2\epsilon_0}\left[\int_{-L/2}^{L/2}\sqrt{R^2+(z-z')^2}\,dz' - z\underbrace{\int_{-L/2}^{L/2}dz'}_{=L} + \underbrace{\int_{-L/2}^{L/2}z'\,dz'}_{=0}\right]. \end{aligned}$$

Make another substitution.

$$z-z' = R\tan\alpha \qquad \rightarrow \qquad R^2+(z-z')^2 = R^2\sec^2\alpha$$

$$-dz' = R\sec^2\alpha\,d\alpha \qquad \rightarrow \qquad dz' = -R\sec^2\alpha\,d\alpha$$

So then

$$\begin{aligned} V(0,0,z) &= \frac{\rho}{2\epsilon_0}\left[-Lz + \int_{\tan^{-1}\left(\frac{z+L/2}{R}\right)}^{\tan^{-1}\left(\frac{z-L/2}{R}\right)}\sqrt{R^2\sec^2\alpha}\,(-R\sec^2\alpha\,d\alpha)\right] \\ &= \frac{\rho}{2\epsilon_0}\left[-Lz + R^2\int_{\tan^{-1}\left(\frac{z-L/2}{R}\right)}^{\tan^{-1}\left(\frac{z+L/2}{R}\right)}\sec^3\alpha\,d\alpha\right]. \end{aligned} \tag{1}$$

Use integration by parts to determine the antiderivative of $\sec^3\alpha$, setting the integration constant to zero.

$$\begin{aligned}
\int \sec^3\alpha\, d\alpha &= \int \sec\alpha(\sec^2\alpha)\, d\alpha \\
&= \int \sec\alpha \frac{d}{d\alpha}(\tan\alpha)\, d\alpha \\
&= \sec\alpha\tan\alpha - \int \frac{d}{d\alpha}(\sec\alpha)\tan\alpha\, d\alpha \\
&= \sec\alpha\tan\alpha - \int (\sec\alpha\tan\alpha)\tan\alpha\, d\alpha \\
&= \sec\alpha\tan\alpha - \int \sec\alpha\tan^2\alpha\, d\alpha \\
&= \sec\alpha\tan\alpha - \int \sec\alpha(\sec^2\alpha - 1)\, d\alpha \\
&= \sec\alpha\tan\alpha - \int \sec^3\alpha\, d\alpha + \int \sec\alpha\, d\alpha \\
2\int \sec^3\alpha\, d\alpha &= \sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha| \\
\int \sec^3\alpha\, d\alpha &= \frac{1}{2}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)
\end{aligned}$$

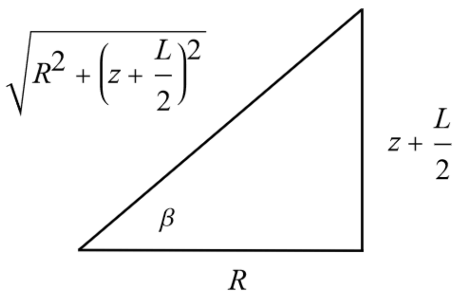
Consequently, equation (1) becomes

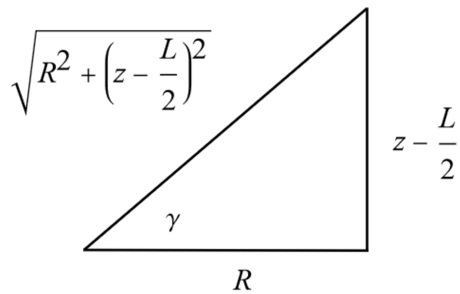
$$\begin{aligned}
V(0,0,z) &= \frac{\rho}{2\epsilon_0}\left[-Lz + \frac{R^2}{2}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)\Big|_{\tan^{-1}\left(\frac{z-L/2}{R}\right)}^{\tan^{-1}\left(\frac{z+L/2}{R}\right)}\right] \\
&= \frac{\rho}{2\epsilon_0}\left\{-Lz + \frac{R^2}{2}\left[\sec\tan^{-1}\left(\frac{z+L/2}{R}\right)\tan\tan^{-1}\left(\frac{z+L/2}{R}\right)\right.\right. \\
&\qquad - \sec\tan^{-1}\left(\frac{z-L/2}{R}\right)\tan\tan^{-1}\left(\frac{z-L/2}{R}\right) \\
&\qquad \left.\left. + \ln\left|\frac{\sec\tan^{-1}\left(\frac{z+L/2}{R}\right) + \tan\tan^{-1}\left(\frac{z+L/2}{R}\right)}{\sec\tan^{-1}\left(\frac{z-L/2}{R}\right) + \tan\tan^{-1}\left(\frac{z-L/2}{R}\right)}\right|\right]\right\}. \qquad (2)
\end{aligned}$$

Let

$$\beta = \tan^{-1}\left(\frac{z+L/2}{R}\right) \quad \text{and} \quad \gamma = \tan^{-1}\left(\frac{z-L/2}{R}\right)$$

and draw the implied triangles in order to determine $\sec\beta$, $\tan\beta$, $\sec\gamma$, and $\tan\gamma$.





$$\sec\beta = \frac{\sqrt{R^2 + (z+L/2)^2}}{R} \qquad \sec\gamma = \frac{\sqrt{R^2 + (z-L/2)^2}}{R}$$

$$\tan\beta = \frac{z+L/2}{R} \qquad \tan\gamma = \frac{z-L/2}{R}$$

As a result, equation (2) becomes

$$\begin{aligned}
V(0,0,z) &= \frac{\rho}{2\epsilon_0}\left[-Lz + \frac{R^2}{2}\left(\sec\beta\tan\beta - \sec\gamma\tan\gamma + \ln\left|\frac{\sec\beta+\tan\beta}{\sec\gamma+\tan\gamma}\right|\right)\right] \\
&= \frac{\rho}{2\epsilon_0}\left\{-Lz + \frac{R^2}{2}\left[\frac{\sqrt{R^2+(z+L/2)^2}}{R}\frac{z+L/2}{R}\right.\right. \\
&\qquad - \frac{\sqrt{R^2+(z-L/2)^2}}{R}\frac{z-L/2}{R} \\
&\qquad \left.\left. + \ln\left|\frac{\frac{\sqrt{R^2+(z+L/2)^2}}{R} + \frac{z+L/2}{R}}{\frac{\sqrt{R^2+(z-L/2)^2}}{R} + \frac{z-L/2}{R}}\right|\right]\right\} \\
&= \frac{\rho}{2\epsilon_0}\left\{-Lz + \frac{1}{2}\left[\left(z+\frac{L}{2}\right)\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}\right.\right. \\
&\qquad - \left(z-\frac{L}{2}\right)\sqrt{R^2+\left(z-\frac{L}{2}\right)^2} \\
&\qquad \left.\left. + R^2\ln\frac{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}+\left(z+\frac{L}{2}\right)}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}+\left(z-\frac{L}{2}\right)}\right]\right\}.
\end{aligned}$$

Therefore, the electric potential on the axis of a uniformly charged solid cylinder, a distance z from the center, is

$$V(0,0,z) = \frac{\rho}{4\epsilon_0}\left[-2Lz + \left(z+\frac{L}{2}\right)\sqrt{R^2+\left(z+\frac{L}{2}\right)^2} - \left(z-\frac{L}{2}\right)\sqrt{R^2+\left(z-\frac{L}{2}\right)^2} + R^2 \ln \frac{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}+\left(z+\frac{L}{2}\right)}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}+\left(z-\frac{L}{2}\right)}\right].$$

Only the z-component of the electric field can be found using

$$\mathbf{E} = -\nabla V$$

because only $V(0,0,z)$ is known. However, due to the symmetry about the z-axis, the x- and y-components of the electric field are zero.

$$\begin{aligned}\mathbf{E} &= -\frac{d}{dz}V(0,0,z)\hat{\mathbf{z}} \\ &= \frac{\rho}{2\epsilon_0}\left[L - \sqrt{R^2+\left(z+\frac{L}{2}\right)^2} + \sqrt{R^2+\left(z-\frac{L}{2}\right)^2}\right]\hat{\mathbf{z}}\end{aligned}$$