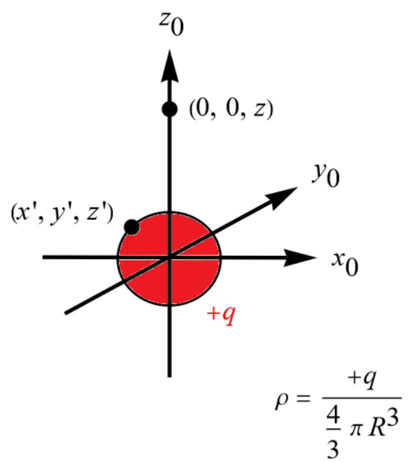
Problem 2.29

Use Eq. 2.29 to calculate the potential inside a uniformly charged solid sphere of radius R and total charge q. Compare your answer to Prob. 2.22.

Solution

Let the origin of the coordinate system be at the center of the solid ball. Because the charge distribution is spherically symmetric, there's freedom to orient the coordinate axes so that the point we want to know the electric potential lies on the polar axis.



$(0, 0, z)$ is where we want to know the electric potential, and (x', y', z') is a point on the charged body. Begin with the basic formula for the electric potential of a continuous volume charge distribution.

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau' \\ &= \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|}\, d\tau' \end{aligned}$$

For the point $(0, 0, z)$ in particular,

$$\begin{aligned} V(0,0,z) &= \frac{1}{4\pi\epsilon_0} \iiint_{x'^2+y'^2+z'^2 \le R^2} \frac{\rho}{|\langle 0, 0, z\rangle - \langle x', y', z'\rangle|}\, dx'\, dy'\, dz' \\ &= \frac{\rho}{4\pi\epsilon_0} \iiint_{x'^2+y'^2+z'^2 \le R^2} \frac{1}{|\langle -x', -y', z - z'\rangle|}\, dx'\, dy'\, dz' \\ &= \frac{\rho}{4\pi\epsilon_0} \iiint_{x'^2+y'^2+z'^2 \le R^2} \frac{1}{\sqrt{x'^2 + y'^2 + (z - z')^2}}\, dx'\, dy'\, dz'. \end{aligned}$$

Switch to spherical coordinates.

$$x' = r'\sin\theta'\cos\phi'$$
$$y' = r'\sin\theta'\sin\phi'$$
$$z' = r'\cos\theta'$$

Consequently,

$$\begin{aligned}
V(0,0,z) &= \frac{\rho}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\frac{1}{\sqrt{(r'\sin\theta'\cos\phi')^2+(r'\sin\theta'\sin\phi')^2+(z-r'\cos\theta')^2}}(r'^2\sin\theta'\,dr'\,d\phi'\,d\theta') \\
&= \frac{\rho}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\frac{1}{\sqrt{r'^2-2r'z\cos\theta'+z^2}}(r'^2\sin\theta'\,dr'\,d\phi'\,d\theta') \\
&= \frac{\rho}{4\pi\epsilon_0}\left(\int_0^{2\pi}d\phi'\right)\int_0^R r'\left[\int_0^{\pi}\frac{r'\sin\theta'}{\sqrt{r'^2-2r'z\cos\theta'+z^2}}\,d\theta'\right]dr'.
\end{aligned}$$

Make the following substitution.

$$u = r'^2-2r'z\cos\theta'+z^2$$
$$du = 2r'z\sin\theta'\,d\theta' \quad\rightarrow\quad \frac{du}{2z} = r'\sin\theta'\,d\theta'$$

As a result,

$$\begin{aligned}
V(0,0,z) &= \frac{\rho}{4\pi\epsilon_0}\left(\int_0^{2\pi}d\phi'\right)\int_0^R r'\left[\int_{r'^2-2r'z\cos 0+z^2}^{r'^2-2r'z\cos\pi+z^2}\frac{1}{\sqrt{u}}\left(\frac{du}{2z}\right)\right]dr' \\
&= \frac{\rho}{4\pi\epsilon_0 z}(2\pi)\int_0^R r'\left(\int_{r'^2-2r'z+z^2}^{r'^2+2r'z+z^2}\frac{1}{2\sqrt{u}}\,du\right)dr' \\
&= \frac{\rho}{2\epsilon_0 z}\int_0^R r'\left[\int_{(r'-z)^2}^{(r'+z)^2}\frac{1}{2\sqrt{u}}\,du\right]dr' \\
&= \frac{\rho}{2\epsilon_0 z}\int_0^R r'\left(\sqrt{u}\right)\Big|_{(r'-z)^2}^{(r'+z)^2}dr' \\
&= \frac{\rho}{2\epsilon_0 z}\int_0^R r'\left[\sqrt{(r'+z)^2}-\sqrt{(r'-z)^2}\right]dr' \\
&= \frac{\rho}{2\epsilon_0 z}\int_0^R r'\left[(r'+z)-|r'-z|\right]dr' \\
&= \begin{cases}\dfrac{\rho}{2\epsilon_0 z}\left\{\displaystyle\int_0^z r'\left[(r'+z)-(z-r')\right]dr'+\int_z^R r'\left[(r'+z)-(r'-z)\right]dr'\right\} & \text{if } z<R \\ \dfrac{\rho}{2\epsilon_0 z}\displaystyle\int_0^R r'\left[(r'+z)-(z-r')\right]dr' & \text{if } z>R\end{cases}.
\end{aligned}$$

Evaluate the integrals and simplify.

$$V(0,0,z) = \begin{cases} \dfrac{\rho}{2\epsilon_0 z}\left[\displaystyle\int_0^z (2r'^2)\,dr' + 2z\int_z^R r'\,dr'\right] & \text{if } z < R \\ \dfrac{\rho}{2\epsilon_0 z}\displaystyle\int_0^R (2r'^2)\,dr' & \text{if } z > R \end{cases}$$

$$= \begin{cases} \dfrac{\rho}{2\epsilon_0 z}\left[\left(\dfrac{2r'^3}{3}\right)\Bigg|_0^z + 2z\left(\dfrac{r'^2}{2}\right)\Bigg|_z^R\right] & \text{if } z < R \\ \dfrac{\rho}{2\epsilon_0 z}\left(\dfrac{2r'^3}{3}\right)\Bigg|_0^R & \text{if } z > R \end{cases}$$

$$= \begin{cases} \dfrac{\rho}{2\epsilon_0 z}\left[\left(\dfrac{2z^3}{3}\right) + 2z\left(\dfrac{R^2}{2} - \dfrac{z^2}{2}\right)\right] & \text{if } z < R \\ \dfrac{\rho}{2\epsilon_0 z}\left(\dfrac{2R^3}{3}\right) & \text{if } z > R \end{cases}$$

$$= \begin{cases} \dfrac{\rho}{2\epsilon_0 z}\left(zR^2 - \dfrac{z^3}{3}\right) & \text{if } z < R \\ \dfrac{\rho R^3}{3\epsilon_0 z} & \text{if } z > R \end{cases}$$

$$= \begin{cases} \dfrac{\rho}{2\epsilon_0 z}\dfrac{zR^2}{3}\left(3 - \dfrac{z^2}{R^2}\right) & \text{if } z < R \\ \dfrac{\rho R^3}{3\epsilon_0 z} & \text{if } z > R \end{cases}$$

$$= \begin{cases} \left(\dfrac{q}{\frac{4}{3}\pi R^3}\right)\dfrac{1}{2\epsilon_0 z}\dfrac{zR^2}{3}\left(3 - \dfrac{z^2}{R^2}\right) & \text{if } z < R \\ \left(\dfrac{q}{\frac{4}{3}\pi R^3}\right)\dfrac{R^3}{3\epsilon_0 z} & \text{if } z > R \end{cases}.$$

Therefore, the electric potential inside ($z < R$) and outside ($z > R$) the uniformly charged solid ball is

$$V(0,0,z) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{2R}\left[3 - \left(\dfrac{z}{R}\right)^2\right] & \text{if } z < R \\ \dfrac{q}{4\pi\epsilon_0 z} & \text{if } z > R \end{cases}.$$

This is the same answer obtained in Problem 2.22 but with z instead of r.