Problem 2.4

Find the electric field a distance z above the center of a square loop (side length a) carrying a uniform line charge λ (Fig. 2.8). [*Hint:* Use the result of Ex. 2.2.]

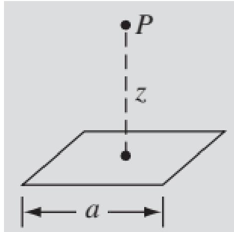
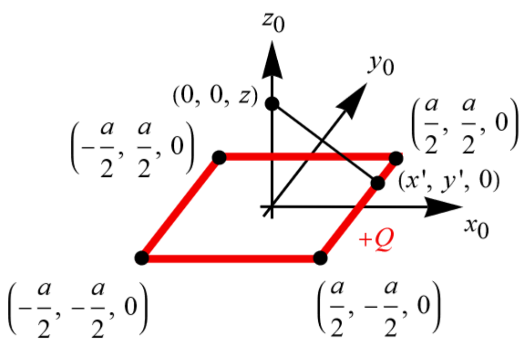


Fig. 2.8

Solution

Start by drawing a schematic for some point on the square loop.

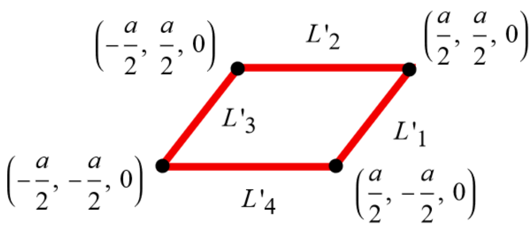


$$\lambda = \frac{+Q}{a+a+a+a} = \frac{Q}{4a}$$

The formula for the electric field from a continuous distribution of charge along a line is

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)dl' \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dl',\end{aligned}$$

where the integral is taken over the line where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose on the line, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the line to where we want to know the electric field. Split up the loop into the four straight line segments, L_1' and L_2' and L_3' and L_4', shown below.



The parameterizations for these lines are as follows.

$$\text{On } L_1': \quad \mathbf{r}' = \left\langle \frac{a}{2}, t, 0 \right\rangle, \qquad -\frac{a}{2} \le t \le \frac{a}{2}$$

$$\text{On } L_2': \quad \mathbf{r}' = \left\langle t, \frac{a}{2}, 0 \right\rangle, \qquad -\frac{a}{2} \le t \le \frac{a}{2}$$

$$\text{On } L_3': \quad \mathbf{r}' = \left\langle -\frac{a}{2}, t, 0 \right\rangle, \qquad -\frac{a}{2} \le t \le \frac{a}{2}$$

$$\text{On } L_4': \quad \mathbf{r}' = \left\langle t, -\frac{a}{2}, 0 \right\rangle, \qquad -\frac{a}{2} \le t \le \frac{a}{2}$$

Consequently, the electric field at $\mathbf{r} = \langle 0, 0, z \rangle$ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[\int_{L_1'} \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dy' + \int_{L_2'} \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dx' + \int_{L_3'} \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dy' + \int_{L_4'} \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dx'\right]$$

$$\begin{aligned}
&= \frac{1}{4\pi\epsilon_0}\left[\int_{L_1'} \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\,dy'\right. \\
&\qquad + \int_{L_2'} \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\,dx' \\
&\qquad + \int_{L_3'} \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\,dy' \\
&\qquad \left. + \int_{L_4'} \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\,dx'\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\pi\epsilon_0}\left\{\int_{-a/2}^{a/2} \frac{\lambda}{\left[\sqrt{\left(0-\frac{a}{2}\right)^2+(0-t)^2+(z-0)^2}\right]^3}\left(\langle 0,0,z\rangle - \left\langle \frac{a}{2}, t, 0\right\rangle\right)\frac{dy'}{dt}\,dt\right. \\
&\qquad + \int_{-a/2}^{a/2} \frac{\lambda}{\left[\sqrt{(0-t)^2+\left(0-\frac{a}{2}\right)^2+(z-0)^2}\right]^3}\left(\langle 0,0,z\rangle - \left\langle t, \frac{a}{2}, 0\right\rangle\right)\frac{dx'}{dt}\,dt \\
&\qquad + \int_{-a/2}^{a/2} \frac{\lambda}{\left[\sqrt{\left(0+\frac{a}{2}\right)^2+(0-t)^2+(z-0)^2}\right]^3}\left(\langle 0,0,z\rangle - \left\langle -\frac{a}{2}, t, 0\right\rangle\right)\frac{dy'}{dt}\,dt \\
&\qquad \left. + \int_{-a/2}^{a/2} \frac{\lambda}{\left[\sqrt{(0-t)^2+\left(0+\frac{a}{2}\right)^2+(z-0)^2}\right]^3}\left(\langle 0,0,z\rangle - \left\langle t, -\frac{a}{2}, 0\right\rangle\right)\frac{dx'}{dt}\,dt\right\}.
\end{aligned}$$

Simplify the integrands, combine the integrals, add the vectors, and then integrate the components.

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\int_{-a/2}^{a/2}\frac{1}{\left(\frac{a^2}{4}+t^2+z^2\right)^{3/2}}\left\langle -\frac{a}{2}, -t, z\right\rangle (1)\,dt + \int_{-a/2}^{a/2}\frac{1}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\left\langle -t, -\frac{a}{2}, z\right\rangle (1)\,dt \right.$$

$$\left. + \int_{-a/2}^{a/2}\frac{1}{\left(\frac{a^2}{4}+t^2+z^2\right)^{3/2}}\left\langle \frac{a}{2}, -t, z\right\rangle (1)\,dt + \int_{-a/2}^{a/2}\frac{1}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\left\langle -t, \frac{a}{2}, z\right\rangle (1)\,dt\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\int_{-a/2}^{a/2}\frac{1}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\left(\left\langle -\frac{a}{2}, -t, z\right\rangle + \left\langle -t, -\frac{a}{2}, z\right\rangle + \left\langle \frac{a}{2}, -t, z\right\rangle + \left\langle -t, \frac{a}{2}, z\right\rangle\right) dt\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{1}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\langle -2t, -2t, 4z\rangle\,dt$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left\langle \int_{-a/2}^{a/2}\frac{-2t\,dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}, \int_{-a/2}^{a/2}\frac{-2t\,dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}, \int_{-a/2}^{a/2}\frac{4z\,dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\right\rangle$$

The first two integrals are zero because the integration interval is symmetric and the integrands are odd functions. The third integral can be made to go from 0 to $a/2$ by putting a factor of 2 in front because the integrand is an even function.

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0}\left\langle 0, 0, 2\int_0^{a/2}\frac{4z\,dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}\right\rangle$$

$$= \frac{\lambda}{4\pi\epsilon_0}\langle 0,0,1\rangle 8z\int_0^{a/2}\frac{dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}$$

$$= \frac{2\lambda z}{\pi\epsilon_0}\hat{\mathbf{z}}\int_0^{a/2}\frac{dt}{\left(t^2+\frac{a^2}{4}+z^2\right)^{3/2}}$$

Make the following substitution.

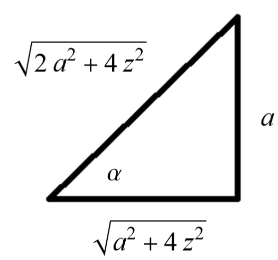
$$t = \sqrt{\frac{a^2}{4}+z^2}\tan\theta \quad \rightarrow \quad t^2+\frac{a^2}{4}+z^2 = \left(\frac{a^2}{4}+z^2\right)(\tan^2\theta+1) = \left(\frac{a^2}{4}+z^2\right)\sec^2\theta$$

$$dt = \sqrt{\frac{a^2}{4}+z^2}\sec^2\theta\,d\theta$$

As a result,

$$\begin{aligned}
\mathbf{E} &= \frac{2\lambda z}{\pi\epsilon_0}\hat{\mathbf{z}}\int_{\tan^{-1}\left(\frac{0}{\sqrt{\frac{a^2}{4}+z^2}}\right)}^{\tan^{-1}\left(\frac{a/2}{\sqrt{\frac{a^2}{4}+z^2}}\right)} \frac{\sqrt{\frac{a^2}{4}+z^2}\sec^2\theta\,d\theta}{\left[\left(\frac{a^2}{4}+z^2\right)\sec^2\theta\right]^{3/2}} \\
&= \frac{2\lambda z}{\pi\epsilon_0}\hat{\mathbf{z}}\int_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+4z^2}}\right)} \frac{\sqrt{\frac{a^2}{4}+z^2}\sec^2\theta\,d\theta}{\left(\frac{a^2}{4}+z^2\right)^{3/2}\sec^3\theta} \\
&= \frac{2\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{\frac{a^2}{4}+z^2}\int_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+4z^2}}\right)} \cos\theta\,d\theta \\
&= \frac{2\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{\frac{a^2}{4}+z^2}\sin\theta\Big|_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+4z^2}}\right)} \\
&= \frac{8\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{a^2+4z^2}\sin\tan^{-1}\left(\frac{a}{\sqrt{a^2+4z^2}}\right).
\end{aligned}$$

Draw the triangle implied by $\alpha = \tan^{-1}\left(a/\sqrt{a^2+4z^2}\right)$ and use it to determine $\sin\alpha$.



$$\sin\alpha = \frac{a}{\sqrt{2a^2+4z^2}}$$

Therefore, the electric field at $\mathbf{r} = \langle 0, 0, z\rangle$ is

$$\boxed{\mathbf{E} = \frac{8\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{a^2+4z^2}\left(\frac{a}{\sqrt{2a^2+4z^2}}\right).}$$

Observe that

$$\begin{aligned}
\lim_{a\to 0}\mathbf{E} &= \lim_{a\to 0}\frac{8\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{a^2+4z^2}\left(\frac{a}{\sqrt{2a^2+4z^2}}\right) = \frac{8\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{(0)^2+4z^2}\left(\frac{0}{\sqrt{2(0)^2+4z^2}}\right) = \mathbf{0} \\
\lim_{z\to 0}\mathbf{E} &= \lim_{z\to 0}\frac{8\lambda z}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{a^2+4z^2}\left(\frac{a}{\sqrt{2a^2+4z^2}}\right) = \frac{8\lambda(0)}{\pi\epsilon_0}\frac{\hat{\mathbf{z}}}{a^2+4(0)^2}\left(\frac{a}{\sqrt{2a^2+4(0)^2}}\right) = \mathbf{0}.
\end{aligned}$$

In order to see what happens if $z \gg a$, rewrite the formula so that each term is a ratio of a and z, z being in the denominator, and use the binomial theorem.

$$\begin{aligned}
\mathbf{E} &= \frac{8\lambda z}{\pi\epsilon_0} \frac{\hat{\mathbf{z}}}{4z^2\left(\frac{a^2}{4z^2}+1\right)} \left(\frac{a}{2z\sqrt{\frac{a^2}{2z^2}+1}}\right) \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \frac{1}{1-\left(-\frac{a^2}{4z^2}\right)} \left(1+\frac{a^2}{2z^2}\right)^{-1/2} \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left[\sum_{k=0}^{\infty} \left(-\frac{a^2}{4z^2}\right)^k\right] \left[\sum_{k=0}^{\infty} \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k+1\right)} \left(\frac{a^2}{2z^2}\right)^k\right] \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left[\sum_{k=0}^{\infty} \left(-\frac{a^2}{4z^2}\right)^k\right] \left[\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}-k\right)} \left(\frac{a^2}{2z^2}\right)^k\right] \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left[\left(-\frac{a^2}{4z^2}\right)^0 + \left(-\frac{a^2}{4z^2}\right)^1 + \left(-\frac{a^2}{4z^2}\right)^2 + \cdots\right] \\
&\qquad\qquad \times \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(\frac{1}{2}\right)} \left(\frac{a^2}{2z^2}\right)^0 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{1}{2}\right)} \left(\frac{a^2}{2z^2}\right)^1 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{3}{2}\right)} \left(\frac{a^2}{2z^2}\right)^2 + \cdots\right] \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left(1 - \frac{a^2}{4z^2} + \frac{a^4}{16z^4} - \cdots\right) \left[1 - \frac{1}{2}\left(\frac{a^2}{2z^2}\right) + \frac{3}{8}\left(\frac{a^4}{4z^4}\right) - \cdots\right] \\
&= \frac{\lambda a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left(1 - \frac{a^2}{2z^2} + \frac{7a^4}{32z^4} - \cdots\right) \\
&= \left(\frac{Q}{4a}\right) \frac{a\hat{\mathbf{z}}}{\pi\epsilon_0 z^2} \left(1 - \frac{a^2}{2z^2} + \frac{7a^4}{32z^4} - \cdots\right) \\
&= \frac{Q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2} \left(1 - \frac{a^2}{2z^2} + \frac{7a^4}{32z^4} - \cdots\right)
\end{aligned}$$

If $z \gg a$, then a^2/z^2 and all higher-order terms are so much smaller than 1 that they can be neglected.

$$\mathbf{E} \approx \frac{Q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{z^2} \hat{\mathbf{z}}$$

The lesson is that far away from the square loop the electric field is the same as if it were a point charge.