Vector Identity (10)

$$\nabla \times (\nabla f) = \mathbf{0}$$

Proof

$$\begin{aligned}
\nabla \times (\nabla f) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) f\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial f}{\partial x_j}\right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i.\text{)} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_j \partial x_i} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k (-\varepsilon_{ijk}) \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \mathbf{0}
\end{aligned}$$