Vector Identity (5)

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

Proof

$$\begin{aligned}
\nabla \cdot (f\mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[f\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j f\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} A_j f \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \left(\frac{\partial A_j}{\partial x_i} f + A_j \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i} f + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_j \frac{\partial f}{\partial x_i} \\
&= f\left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i}\right] + \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) A_i \frac{\partial f}{\partial x_j} \\
&= f\left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right)\right] + \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i A_i\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial f}{\partial x_j}\right) \\
&= f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f
\end{aligned}$$