Problem A.4

Let $f(y) = \int_0^2 \delta(y - x(2-x))\,dx$. Find $f(y)$, and plot it from $y = -2$ to $y = +2$.

Solution

Use Equation A.13 on page 426,

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x - x_i), \tag{A.13}$$

to simplify the integrand. Let $g(x) = y - x(2-x)$, treating y as a constant, and find the zeros.

$$y - x(2-x) = 0$$

$$x^2 - 2x + y = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4y}}{2}$$

$$x = 1 \pm \sqrt{1-y}$$

Evaluate the derivative of $g(x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}(y - x(2-x)) \\ &= 2x - 2 \end{aligned}$$

Consequently, by Equation A.13,

$$\delta(y - x(2-x)) = \begin{cases} \frac{1}{|g'(1-\sqrt{1-y})|}\delta(x - 1 + \sqrt{1-y}) + \frac{1}{|g'(1+\sqrt{1-y})|}\delta(x - 1 - \sqrt{1-y}) & \text{if } -2 \le y < 1 \\ \frac{1}{|g'(1)|}\delta(x-1) & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{1-y}}\delta(x - 1 + \sqrt{1-y}) + \frac{1}{2\sqrt{1-y}}\delta(x - 1 - \sqrt{1-y}) & \text{if } -2 \le y < 1 \\ \frac{1}{0}\delta(x-1) & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{1-y}}\left[\delta(x - 1 + \sqrt{1-y}) + \delta(x - 1 - \sqrt{1-y})\right] & \text{if } -2 \le y < 1 \\ \text{undefined} & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases} .$$

Integrate both sides with respect to x from 0 to 2.

$$\int_0^2 \delta(y - x(2-x))\,dx = \begin{cases} \dfrac{1}{2\sqrt{1-y}}\left[\displaystyle\int_0^2 \delta(x-1+\sqrt{1-y})\,dx + \int_0^2 \delta(x-1-\sqrt{1-y})\,dx\right] & \text{if } -2 \le y < 1 \\ \text{undefined} & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}$$

The first integral is zero if $1-\sqrt{1-y}$ is outside the interval $(0,2)$ and one if it's inside the interval. The second integral is zero if $1+\sqrt{1-y}$ is outside the interval $(0,2)$ and one if it's inside the interval.

$$= \begin{cases} \dfrac{1}{2\sqrt{1-y}}\left[\overbrace{\displaystyle\int_0^2 \delta(x-1+\sqrt{1-y})\,dx}^{=0} + \overbrace{\int_0^2 \delta(x-1-\sqrt{1-y})\,dx}^{=0}\right] & \text{if } -2 \le y \le 0 \\ \dfrac{1}{2\sqrt{1-y}}\left[\underbrace{\displaystyle\int_0^2 \delta(x-1+\sqrt{1-y})\,dx}_{=1} + \underbrace{\int_0^2 \delta(x-1-\sqrt{1-y})\,dx}_{=1}\right] & \text{if } 0 < y < 1 \\ \text{undefined} & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{1-y}}(0) & \text{if } -2 \le y \le 0 \\ \dfrac{1}{2\sqrt{1-y}}(2) & \text{if } 0 < y < 1 \\ \text{undefined} & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}$$

Therefore,

$$f(y) = \int_0^2 \delta(y - x(2-x))\,dx = \begin{cases} 0 & \text{if } -2 \le y \le 0 \\ \dfrac{1}{\sqrt{1-y}} & \text{if } 0 < y < 1 \\ \text{undefined} & \text{if } y = 1 \\ 0 & \text{if } 1 \le y \le 2 \end{cases}.$$

Below is a plot of $f(y)$ versus y for $-2 \leq y \leq 2$.