

Problem A.5

$$\int_{-1}^5 x^4 \left[\frac{d^2}{dx^2} \delta(x-3) \right] dx = ? \text{ [Hint: Integrate by parts.]}$$

Solution

Integrate by parts twice, noting that $\delta(x-3)$ and the slope of $\delta(x-3)$ are zero away from $x=3$.

$$\begin{aligned} \int_{-1}^5 x^4 \left[\frac{d^2}{dx^2} \delta(x-3) \right] dx &= \underbrace{x^4 \left[\frac{d}{dx} \delta(x-3) \right] \Big|_{-1}^5}_{=0} - \int_{-1}^5 \frac{d}{dx} (x^4) \left[\frac{d}{dx} \delta(x-3) \right] dx \\ &= - \int_{-1}^5 (4x^3) \left[\frac{d}{dx} \delta(x-3) \right] dx \\ &= -4 \int_{-1}^5 x^3 \left[\frac{d}{dx} \delta(x-3) \right] dx \\ &= -4 \left[\underbrace{x^3 \delta(x-3) \Big|_{-1}^5}_{=0} - \int_{-1}^5 \frac{d}{dx} (x^3) \delta(x-3) dx \right] \\ &= 4 \int_{-1}^5 (3x^2) \delta(x-3) dx \\ &= 12 \int_{-1}^5 x^2 \delta(x-3) dx \\ &= 12(x^2) \Big|_{x=3} \\ &= 12(3^2) \\ &= 108 \end{aligned}$$