

Problem 1.5

Prove the **BAC-CAB** rule by writing out both sides in component form.

Solution

The BAC-CAB rule is

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

Evaluate the left-hand side.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \mathbf{A} \times [(B_y C_z - B_z C_y)\hat{\mathbf{x}} - (B_x C_z - B_z C_x)\hat{\mathbf{y}} + (B_x C_y - B_y C_x)\hat{\mathbf{z}}] \\ &= \mathbf{A} \times [(B_y C_z - B_z C_y)\hat{\mathbf{x}} + (B_z C_x - B_x C_z)\hat{\mathbf{y}} + (B_x C_y - B_y C_x)\hat{\mathbf{z}}] \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \\ &= [A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)]\hat{\mathbf{x}} \\ &\quad - [A_x(B_x C_y - B_y C_x) - A_z(B_y C_z - B_z C_y)]\hat{\mathbf{y}} \\ &\quad + [A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)]\hat{\mathbf{z}} \\ &= (A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z)\hat{\mathbf{x}} \\ &\quad + (-A_x B_x C_y + A_x B_y C_x + A_z B_y C_z - A_z B_z C_y)\hat{\mathbf{y}} \\ &\quad + (A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y)\hat{\mathbf{z}} \end{aligned}$$

Evaluate the right-hand side.

$$\begin{aligned} \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}})(A_x C_x + A_y C_y + A_z C_z) \\ &\quad - (C_x \hat{\mathbf{x}} + C_y \hat{\mathbf{y}} + C_z \hat{\mathbf{z}})(A_x B_x + A_y B_y + A_z B_z) \\ &= [B_x(A_x C_x + A_y C_y + A_z C_z)\hat{\mathbf{x}} + B_y(A_x C_x + A_y C_y + A_z C_z)\hat{\mathbf{y}} \\ &\quad + B_z(A_x C_x + A_y C_y + A_z C_z)\hat{\mathbf{z}}] - [C_x(A_x B_x + A_y B_y + A_z B_z)\hat{\mathbf{x}} \\ &\quad + C_y(A_x B_x + A_y B_y + A_z B_z)\hat{\mathbf{y}} + C_z(A_x B_x + A_y B_y + A_z B_z)\hat{\mathbf{z}}] \\ &= [B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z)]\hat{\mathbf{x}} \\ &\quad + [B_y(A_x C_x + A_y C_y + A_z C_z) - C_y(A_x B_x + A_y B_y + A_z B_z)]\hat{\mathbf{y}} \\ &\quad + [B_z(A_x C_x + A_y C_y + A_z C_z) - C_z(A_x B_x + A_y B_y + A_z B_z)]\hat{\mathbf{z}} \\ &= (\cancel{A_x B_x C_x} + A_y B_x C_y + A_z B_x C_z - \cancel{A_x B_x C_x} - A_y B_y C_x - A_z B_z C_x)\hat{\mathbf{x}} \\ &\quad + (A_x B_y C_x + \cancel{A_y B_y C_y} + A_z B_y C_z - A_x B_x C_y - \cancel{A_y B_y C_y} - A_z B_z C_y)\hat{\mathbf{y}} \\ &\quad + (A_x B_z C_x + A_y B_z C_y + \cancel{A_z B_z C_z} - A_x B_x C_z - A_y B_y C_z - \cancel{A_z B_z C_z})\hat{\mathbf{z}} \\ &= (A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z)\hat{\mathbf{x}} \\ &\quad + (-A_x B_x C_y + A_x B_y C_x + A_z B_y C_z - A_z B_z C_y)\hat{\mathbf{y}} \\ &\quad + (A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y)\hat{\mathbf{z}} \end{aligned}$$

Both sides have equal components, so the rule is proven.