

Vector Identity (1)

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Proof

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \left(\sum_{i=1}^3 \delta_i A_i \right) \cdot \left[\left(\sum_{j=1}^3 \delta_j B_j \right) \times \left(\sum_{k=1}^3 \delta_k C_k \right) \right] \\
&= \left(\sum_{i=1}^3 \delta_i A_i \right) \cdot \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) B_j C_k \right] \\
&= \left(\sum_{i=1}^3 \delta_i A_i \right) \cdot \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{ljk} B_j C_k \right) \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \cdot \delta_l) \varepsilon_{ljk} A_i B_j C_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{il} \varepsilon_{ljk} A_i B_j C_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} A_i B_j C_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{kij} A_i B_j C_k \tag{1} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{kl} \varepsilon_{lij} A_i B_j C_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_k \cdot \delta_l) \varepsilon_{lij} A_i B_j C_k \\
&= \left(\sum_{k=1}^3 \delta_k C_k \right) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{lij} A_i B_j \right) \\
&= \left(\sum_{k=1}^3 \delta_k C_k \right) \cdot \left[\sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \times \delta_j) A_i B_j \right] \\
&= \left(\sum_{k=1}^3 \delta_k C_k \right) \cdot \left[\left(\sum_{i=1}^3 \delta_i A_i \right) \times \left(\sum_{j=1}^3 \delta_j B_j \right) \right] \\
&= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$

To get the last identity, return to equation (1).

$$\begin{aligned}
 \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{kij} A_i B_j C_k & (1) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{jki} A_i B_j C_k \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{jl} \varepsilon_{lki} A_i B_j C_k \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_j \cdot \delta_l) \varepsilon_{lki} A_i B_j C_k \\
 &= \left(\sum_{j=1}^3 \delta_j B_j \right) \cdot \left(\sum_{i=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{lki} A_i C_k \right) \\
 &= \left(\sum_{j=1}^3 \delta_j B_j \right) \cdot \left[\sum_{i=1}^3 \sum_{k=1}^3 (\delta_k \times \delta_i) A_i C_k \right] \\
 &= \left(\sum_{j=1}^3 \delta_j B_j \right) \cdot \left[\left(\sum_{k=1}^3 \delta_k C_k \right) \times \left(\sum_{i=1}^3 \delta_i A_i \right) \right] \\
 &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})
 \end{aligned}$$

Therefore,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$