

Vector Identity (8)

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Proof

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left[\left(\sum_{j=1}^3 \delta_j A_j \right) \times \left(\sum_{k=1}^3 \delta_k B_k \right) \right] \\
&= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) A_j B_k \right] \\
&= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} A_j B_k \right) \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \times \delta_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}) \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mj} \delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mk} \delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \frac{\partial}{\partial x_k} A_j B_k - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \frac{\partial}{\partial x_j} A_j B_k \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \left(\frac{\partial A_j}{\partial x_k} B_k + A_j \frac{\partial B_k}{\partial x_k} \right) - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \left(\frac{\partial A_j}{\partial x_j} B_k + A_j \frac{\partial B_k}{\partial x_j} \right) \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \frac{\partial A_j}{\partial x_k} B_k + \sum_{j=1}^3 \sum_{k=1}^3 \delta_j A_j \frac{\partial B_k}{\partial x_k} - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \frac{\partial A_j}{\partial x_j} B_k - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k A_j \frac{\partial B_k}{\partial x_j} \\
&= \sum_{k=1}^3 B_k \frac{\partial}{\partial x_k} \left(\sum_{j=1}^3 \delta_j A_j \right) + \left(\sum_{j=1}^3 \delta_j A_j \right) \sum_{k=1}^3 \frac{\partial B_k}{\partial x_k} - \left(\sum_{k=1}^3 \delta_k B_k \right) \sum_{j=1}^3 \frac{\partial A_j}{\partial x_j} - \sum_{j=1}^3 A_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k B_k \right) \\
&= (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}
\end{aligned}$$