Problem A.10

By explicit construction of the matrices in question, show that any matrix T can be written

(a) as the sum of a symmetric matrix S and an antisymmetric matrix A;

(b) as the sum of a real matrix R and an imaginary matrix M;

(c) as the sum of a hermitian matrix H and a skew-hermitian matrix K.

Solution

Part (a)

Here the aim is to find a symmetric matrix S and an antisymmetric matrix A such that

$$\mathsf{T} = \mathsf{S} + \mathsf{A}. \tag{1}$$

Take the transpose of both sides.

$$\begin{aligned} \widetilde{\mathsf{T}} &= \widetilde{\mathsf{S} + \mathsf{A}} \\ &= \widetilde{\mathsf{S}} + \widetilde{\mathsf{A}} \\ &= (+\mathsf{S}) + (-\mathsf{A}) \\ &= \mathsf{S} - \mathsf{A} \end{aligned} \tag{2}$$

Add the respective sides of equations (1) and (2) to eliminate A.

$$\mathsf{T} + \widetilde{\mathsf{T}} = 2\mathsf{S}$$

Solve for S.

$$\mathsf{S} = \frac{\mathsf{T} + \widetilde{\mathsf{T}}}{2}$$

Subtract the respective sides of equations (1) and (2) to eliminate S.

$$\mathsf{T} - \widetilde{\mathsf{T}} = 2\mathsf{A}$$

Solve for A.

$$\mathsf{A} = \frac{\mathsf{T} - \widetilde{\mathsf{T}}}{2}$$

Part (b)

Here the aim is to find a real matrix R and an imaginary matrix M such that

$$\mathsf{T} = \mathsf{R} + \mathsf{M}. \tag{3}$$

Take the complex conjugate of both sides.

$$\begin{aligned}\mathsf{T}^* &= (\mathsf{R} + \mathsf{M})^* \\ &= \mathsf{R}^* + \mathsf{M}^* \\ &= (+\mathsf{R}) + (-\mathsf{M}) \\ &= \mathsf{R} - \mathsf{M} \end{aligned} \tag{4}$$

Add the respective sides of equations (3) and (4) to eliminate M.

$$\mathsf{T} + \mathsf{T}^* = 2\mathsf{R}$$

Solve for R.

$$\mathsf{R} = \frac{\mathsf{T} + \mathsf{T}^*}{2}$$

Subtract the respective sides of equations (3) and (4) to eliminate R.

$$\mathsf{T} - \mathsf{T}^* = 2\mathsf{M}$$

Solve for M.

$$\mathsf{M} = \frac{\mathsf{T} - \mathsf{T}^*}{2}$$

Part (c)

Here the aim is to find a hermitian matrix H and a skew-hermitian matrix K such that

$$\mathsf{T} = \mathsf{H} + \mathsf{K}. \tag{5}$$

Take the hermitian conjugate of both sides.

$$\begin{aligned}\mathsf{T}^\dagger &= (\mathsf{H} + \mathsf{K})^\dagger \\ &= \mathsf{H}^\dagger + \mathsf{K}^\dagger \\ &= (+\mathsf{H}) + (-\mathsf{K}) \\ &= \mathsf{H} - \mathsf{K} \end{aligned} \tag{6}$$

Add the respective sides of equations (5) and (6) to eliminate K.

$$\mathsf{T} + \mathsf{T}^\dagger = 2\mathsf{H}$$

Solve for H.

$$\mathsf{H} = \frac{\mathsf{T} + \mathsf{T}^\dagger}{2}$$

Subtract the respective sides of equations (5) and (6) to eliminate H.

$$\mathsf{T} - \mathsf{T}^\dagger = 2\mathsf{K}$$

Solve for K.

$$\mathsf{K} = \frac{\mathsf{T} - \mathsf{T}^\dagger}{2}$$