Problem A.11

Prove Equations A.52, A.53, and A.58. Show that the product of two unitary matrices is unitary. Under what conditions is the product of two hermitian matrices hermitian? Is the sum of two unitary matrices necessarily unitary? Is the sum of two hermitian matrices always hermitian?

Solution

In order to prove the following matrix equations, one needs to show that the corresponding elements on both sides are equal.

Equation A.52

Equation A.52 involves the transpose of a matrix product.

$$\widetilde{\mathsf{ST}} = \widetilde{\mathsf{T}}\widetilde{\mathsf{S}} \tag{A.52}$$

Consider element ij of the matrix on the left side.

$$\begin{aligned} \left(\widetilde{\mathsf{ST}}\right)_{ij} &= (\mathsf{ST})_{ji} \\ &= \sum_{k=1}^{n} S_{jk}T_{ki} \\ &= \sum_{k=1}^{n} T_{ki}S_{jk} \\ &= \sum_{k=1}^{n} \widetilde{T}_{ik}\widetilde{S}_{kj} \\ &= \left(\widetilde{\mathsf{T}}\widetilde{\mathsf{S}}\right)_{ij} \end{aligned}$$

Therefore,

$$\widetilde{\mathsf{ST}} = \widetilde{\mathsf{T}}\widetilde{\mathsf{S}}.$$

Equation A.53

Equation A.53 involves the hermitian conjugate of a matrix product.

$$(\mathsf{ST})^\dagger = \mathsf{T}^\dagger \mathsf{S}^\dagger \tag{A.53}$$

Consider element ij of the matrix on the left side.

$$\begin{aligned}
\left[(\mathsf{ST})^\dagger\right]_{ij} &= \left[\widetilde{(\mathsf{ST})^*}\right]_{ij} \\
&= [(\mathsf{ST})^*]_{ji} \\
&= \left(\sum_{k=1}^n S_{jk}T_{ki}\right)^* \\
&= \sum_{k=1}^n (S_{jk}T_{ki})^* \\
&= \sum_{k=1}^n S_{jk}^* T_{ki}^* \\
&= \sum_{k=1}^n T_{ki}^* S_{jk}^* \\
&= \sum_{k=1}^n \widetilde{T}_{ik}^* \widetilde{S}_{kj}^* \\
&= \sum_{k=1}^n T_{ik}^\dagger S_{kj}^\dagger \\
&= \left(\mathsf{T}^\dagger \mathsf{S}^\dagger\right)_{ij}
\end{aligned}$$

Therefore,

$$(\mathsf{ST})^\dagger = \mathsf{T}^\dagger \mathsf{S}^\dagger.$$

Equation A.58

Equation A.58 involves the inverse of a matrix product.

$$(\mathsf{ST})^{-1} = \mathsf{T}^{-1}\mathsf{S}^{-1} \tag{A.58}$$

For an $n \times n$ matrix X, the minor of element X_{ij} is defined to be the determinant of the matrix obtained by removing row i and column j; denote it by M_{ij}^{X}. Also, the cofactor of element X_{ij} is defined to be $C_{ij}^{\mathsf{X}} = (-1)^{i+j} M_{ij}^{\mathsf{X}}$. Consider element ij of the matrix on the left side.

$$\begin{aligned}
\left[(\mathsf{ST})^{-1}\right]_{ij} &= \left[\frac{1}{\det(\mathsf{ST})}\widetilde{\mathsf{C}}^{\mathsf{ST}}\right]_{ij} \\
&= \frac{1}{\det(\mathsf{ST})}\widetilde{C}_{ij}^{\mathsf{ST}} \\
&= \frac{1}{\det(\mathsf{ST})}C_{ji}^{\mathsf{ST}} \\
&= \frac{1}{\det(\mathsf{ST})}(-1)^{j+i}M_{ji}^{\mathsf{ST}}
\end{aligned}$$

The minor of a product is the product of the minors: $\mathsf{M}^{\mathsf{XY}} = \mathsf{M}^{\mathsf{X}}\mathsf{M}^{\mathsf{Y}}$.

$$\begin{aligned}
\left[(\mathsf{ST})^{-1}\right]_{ij} &= \frac{1}{\det(\mathsf{ST})}(-1)^{j+i}\sum_{k=1}^{n} M_{jk}^{\mathsf{S}} M_{ki}^{\mathsf{T}} \\
&= \frac{1}{\det(\mathsf{ST})}(-1)^{j+i}\sum_{k=1}^{n}(-1)^{2k} M_{jk}^{\mathsf{S}} M_{ki}^{\mathsf{T}} \\
&= \frac{1}{\det(\mathsf{ST})}\sum_{k=1}^{n}\left[(-1)^{j+k}M_{jk}^{\mathsf{S}}\right]\left[(-1)^{k+i}M_{ki}^{\mathsf{T}}\right] \\
&= \frac{1}{\det(\mathsf{ST})}\sum_{k=1}^{n} C_{jk}^{\mathsf{S}} C_{ki}^{\mathsf{T}}
\end{aligned}$$

The determinant of a product is the product of the determinants: $\det(\mathsf{XY}) = \det(\mathsf{X})\det(\mathsf{Y})$.

$$\begin{aligned}
\left[(\mathsf{ST})^{-1}\right]_{ij} &= \frac{1}{\det(\mathsf{S})\det(\mathsf{T})}\sum_{k=1}^{n} C_{ki}^{\mathsf{T}} C_{jk}^{\mathsf{S}} \\
&= \frac{1}{\det(\mathsf{S})\det(\mathsf{T})}\sum_{k=1}^{n} \widetilde{C}_{ik}^{\mathsf{T}} \widetilde{C}_{kj}^{\mathsf{S}} \\
&= \left[\frac{1}{\det(\mathsf{S})\det(\mathsf{T})}\widetilde{\mathsf{C}}^{\mathsf{T}}\widetilde{\mathsf{C}}^{\mathsf{S}}\right]_{ij} \\
&= \left\{\left[\frac{1}{\det(\mathsf{T})}\widetilde{\mathsf{C}}^{\mathsf{T}}\right]\left[\frac{1}{\det(\mathsf{S})}\widetilde{\mathsf{C}}^{\mathsf{S}}\right]\right\}_{ij} = (\mathsf{T}^{-1}\mathsf{S}^{-1})_{ij}
\end{aligned}$$

Therefore,

$$(\mathsf{ST})^{-1} = \mathsf{T}^{-1}\mathsf{S}^{-1}.$$

This result can be verified by premultiplying ST by $\mathsf{T}^{-1}\mathsf{S}^{-1}$ and by postmultiplying ST by $\mathsf{T}^{-1}\mathsf{S}^{-1}$ and seeing that the identity matrix is obtained in both cases.

$$\begin{cases} (\mathsf{T}^{-1}\mathsf{S}^{-1})\mathsf{ST} = \mathsf{T}^{-1}(\mathsf{S}^{-1}\mathsf{S})\mathsf{T} = \mathsf{T}^{-1}(\mathsf{I})\mathsf{T} = \mathsf{T}^{-1}(\mathsf{IT}) = \mathsf{T}^{-1}\mathsf{T} = \mathsf{I} \\ \mathsf{ST}(\mathsf{T}^{-1}\mathsf{S}^{-1}) = \mathsf{S}(\mathsf{TT}^{-1})\mathsf{S}^{-1} = \mathsf{S}(\mathsf{I})\mathsf{S}^{-1} = (\mathsf{SI})\mathsf{S}^{-1} = \mathsf{SS}^{-1} = \mathsf{I} \end{cases}$$

Hermitian Matrices

A hermitian matrix is a matrix that's equal to its hermitian conjugate.

$$\mathsf{H} = \mathsf{H}^\dagger$$

Suppose that A and B are hermitian matrices. Check to see if the sum of A and B is hermitian.

$$\begin{aligned} \mathsf{A} + \mathsf{B} &\stackrel{?}{=} (\mathsf{A} + \mathsf{B})^\dagger \\ &\stackrel{?}{=} \mathsf{A}^\dagger + \mathsf{B}^\dagger \\ &= \mathsf{A} + \mathsf{B} \end{aligned}$$

The sum of two hermitian matrices is hermitian. Now check to see if the product of A and B is hermitian.

$$\begin{aligned} \mathsf{AB} &\stackrel{?}{=} (\mathsf{AB})^\dagger \\ &\stackrel{?}{=} \mathsf{B}^\dagger\mathsf{A}^\dagger \\ &\stackrel{?}{=} \mathsf{BA} \end{aligned}$$

If it so happens that $\mathsf{AB} = \mathsf{BA}$, then AB is hermitian; otherwise, the product of two hermitian matrices is not hermitian.

Unitary Matrices

A unitary matrix is a matrix whose inverse is equal to its hermitian conjugate.

$$\mathsf{U}^{-1} = \mathsf{U}^\dagger$$

Suppose that A and B are unitary matrices. Check to see if the sum of A and B is unitary.

$$\begin{aligned}(\mathsf{A}+\mathsf{B})^{-1} &\stackrel{?}{=} (\mathsf{A}+\mathsf{B})^\dagger \\ &\stackrel{?}{=} \mathsf{A}^\dagger + \mathsf{B}^\dagger \\ &\stackrel{?}{=} \mathsf{A}^{-1} + \mathsf{B}^{-1}\end{aligned}$$

If it so happens that $(\mathsf{A}+\mathsf{B})^{-1} = \mathsf{A}^{-1} + \mathsf{B}^{-1}$, then $\mathsf{A} + \mathsf{B}$ is unitary; otherwise, the sum of two unitary matrices is not unitary. Now check to see if the product of A and B is unitary.

$$\begin{aligned}(\mathsf{AB})^{-1} &\stackrel{?}{=} (\mathsf{AB})^\dagger \\ \mathsf{B}^{-1}\mathsf{A}^{-1} &\stackrel{?}{=} \mathsf{B}^\dagger \mathsf{A}^\dagger \\ &= \mathsf{B}^{-1}\mathsf{A}^{-1}\end{aligned}$$

The product of two unitary matrices is unitary.