Problem A.13

Noting that $\det(\widetilde{\mathsf{T}}) = \det(\mathsf{T})$, show that the determinant of a hermitian matrix is real, the determinant of a unitary matrix has modulus 1 (hence the name), and the determinant of an orthogonal matrix (footnote 13) is either $+1$ or -1.

Solution

Note also that

$$\begin{aligned}\det(\mathsf{T}^*) &= \left[\sum \sigma(p_1, p_2, \ldots, p_n) T^*_{1p_1} T^*_{2p_2} \cdots T^*_{np_n}\right] \\ &= \left[\sum \sigma(p_1, p_2, \ldots, p_n) T_{1p_1} T_{2p_2} \cdots T_{np_n}\right]^* \\ &= (\det \mathsf{T})^*,\end{aligned}$$

where the sum is taken over the $n!$ permutations of $1, 2, \ldots, n$, and $\sigma = 1$ or $\sigma = -1$ if $(p_1, p_2, \ldots, p_n)$ has even parity or odd parity, respectively. A hermitian matrix is a matrix that's equal to its hermitian conjugate.

$$\mathsf{H} = \mathsf{H}^\dagger$$

Take the determinant of both sides.

$$\begin{aligned}\det \mathsf{H} &= \det(\mathsf{H}^\dagger) \\ &= \det(\widetilde{\mathsf{H}^*}) \\ &= \det(\mathsf{H}^*) \\ &= (\det \mathsf{H})^*\end{aligned}$$

Therefore, the determinant of a hermitian matrix is real. A unitary matrix is a matrix whose inverse is equal to its hermitian conjugate.

$$\mathsf{U}^{-1} = \mathsf{U}^\dagger$$

Premultiply or postmultiply both sides by U.

$$\mathsf{U}\mathsf{U}^{-1} = \mathsf{U}\mathsf{U}^\dagger$$

The product of a matrix with its inverse is the identity matrix.

$$\mathsf{I} = \mathsf{U}\mathsf{U}^\dagger$$

Take the determinant of both sides.

$$\det \mathsf{I} = \det(\mathsf{U}\mathsf{U}^\dagger)$$

The determinant of the identity matrix is 1. Use the fact that the determinant of a product is the product of the determinants.

$$\begin{aligned}1 &= \det(\mathsf{U})\det(\mathsf{U}^\dagger)\\ &= \det(\mathsf{U})\det(\widetilde{\mathsf{U}^*})\\ &= \det(\mathsf{U})\det(\mathsf{U}^*)\\ &= (\det\mathsf{U})(\det\mathsf{U})^*\\ &= |\det\mathsf{U}|^2\end{aligned}$$

Take the square root of both sides, discarding the minus sign since the modulus must be positive.

$$|\det\mathsf{U}| = 1$$

Therefore, the determinant of a unitary matrix has modulus 1. An orthogonal matrix is a matrix whose inverse is equal to its transpose.

$$\mathsf{O}^{-1} = \widetilde{\mathsf{O}}$$

Premultiply or postmultiply both sides by O.

$$\mathsf{O}\mathsf{O}^{-1} = \mathsf{O}\widetilde{\mathsf{O}}$$

The product of a matrix with its inverse is the identity matrix.

$$\mathsf{I} = \mathsf{O}\widetilde{\mathsf{O}}$$

Take the determinant of both sides.

$$\det\mathsf{I} = \det(\mathsf{O}\widetilde{\mathsf{O}})$$

The determinant of the identity matrix is 1. Use the fact that the determinant of a product is the product of the determinants.

$$\begin{aligned}1 &= \det(\mathsf{O})\det(\widetilde{\mathsf{O}})\\ &= \det(\mathsf{O})\det(\mathsf{O})\\ &= (\det\mathsf{O})^2\end{aligned}$$

Take the square root of both sides.

$$\det\mathsf{O} = \pm 1$$

Therefore, the determinant of an orthogonal matrix is either $+1$ or -1.