Problem A.14

Using the standard basis $\left(\hat{\imath}, \hat{\jmath}, \hat{k}\right)$ for vectors in three dimensions:

(a) Construct the matrix representing a rotation through angle θ (counterclockwise, looking down the axis toward the origin) about the z axis.

(b) Construct the matrix representing a rotation by 120° (counterclockwise, looking down the axis) about an axis through the point $(1, 1, 1)$.

(c) Construct the matrix representing reflection through the xy plane.

(d) Check that all these matrices are orthogonal, and calculate their determinants.

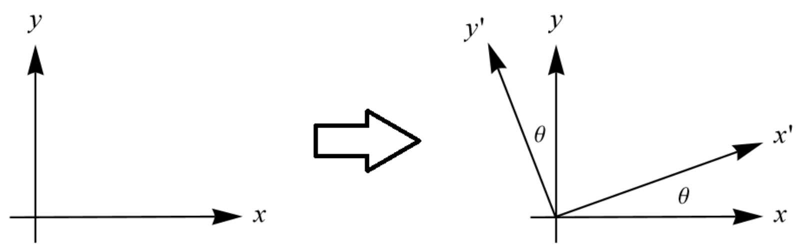
Solution

The goal in parts (a), (b), and (c) is to find T, the matrix representing the prescribed linear transformation with respect to the standard basis. (See Equation A.42 on page 470.)

$$\mathsf{a}' = \mathsf{T}\mathsf{a} \tag{A.42}$$

Part (a)

Draw an xyz-coordinate system, looking at the origin from the positive z-axis.



A counterclockwise rotation about the z-axis of angle θ results in the axes as shown on the right. The new unit vectors are expressed in terms of the old ones as follows.

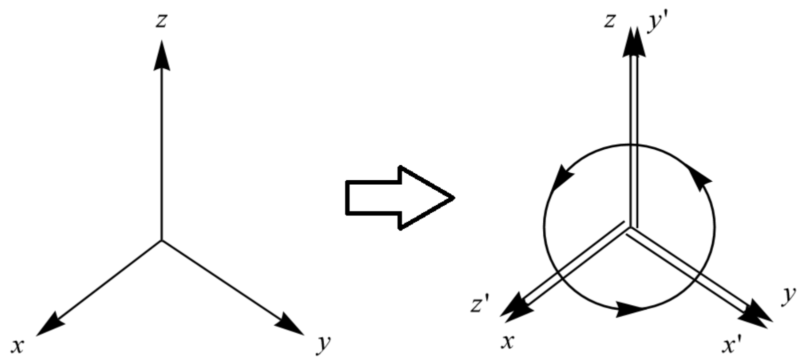
$$\begin{cases} \hat{\imath}' = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \hat{\jmath}' = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \\ \hat{k}' = \hat{k} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix}$$

Therefore,

$$\mathsf{T}_a = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Part (b)

Draw an xyz-coordinate system, looking at the origin from the point $(1, 1, 1)$.



A counterclockwise rotation of 120° about the line of sight through the origin makes the new z-axis point in the direction of the x-axis, makes the new x-axis point in the direction of the y-axis, and makes the new y-axis point in the direction of the z-axis. The new unit vectors are expressed in terms of the old ones as follows.

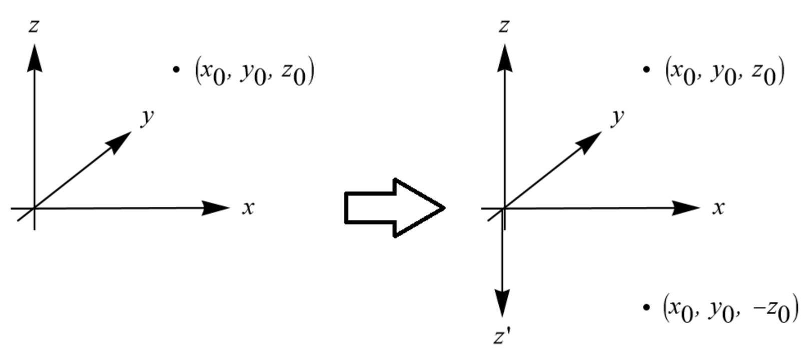
$$\begin{cases} \hat{i}' = \hat{j} \\ \hat{j}' = \hat{k} \\ \hat{k}' = \hat{i} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Therefore,

$$\mathsf{T}_b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Part (c)

Draw an xyz-coordinate system, labelling a particular point above the xy-plane.



The reflection of this point about the xy-plane has the same x- and y- coordinates and a z-coordinate of opposite sign.

The new unit vectors are expressed in terms of the old ones as follows.

$$\begin{cases} \hat{i}' = \hat{i} \\ \hat{j}' = \hat{j} \\ \hat{k}' = -\hat{k} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Therefore,

$$\mathsf{T}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Part (d)

Check that each matrix is orthogonal.

$$\mathsf{T}_a^\dagger \mathsf{T}_a = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{I}$$

$$\mathsf{T}_b^\dagger \mathsf{T}_b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{I}$$

$$\mathsf{T}_c^\dagger \mathsf{T}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{I}$$

Calculate the determinant of each matrix.

$$\det \mathsf{T}_a = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\det \mathsf{T}_b = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$\det \mathsf{T}_c = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 - 0 = -1$$