Problem A.16

Show that similarity preserves matrix multiplication (that is, if $\mathsf{A}^e\mathsf{B}^e = \mathsf{C}^e$, then $\mathsf{A}^f\mathsf{B}^f = \mathsf{C}^f$). Similarity does *not*, in general, preserve symmetry, reality, or hermiticity; show, however, that if S is *unitary*, and H^e is hermitian, then H^f is hermitian. Show that S carries an orthonormal basis into another orthonormal basis if and only if it is unitary.

Solution

Let there be an old orthonormal basis $\langle e_i \,|\, e_j\rangle = \delta_{ij}$ and a new orthonormal basis $\langle f_i \,|\, f_j\rangle = \delta_{ij}$. Suppose that in this old basis two matrices multiply to give $\mathsf{A}^e\mathsf{B}^e = \mathsf{C}^e$. In the new basis the corresponding matrices multiply to give

$$\begin{aligned}\mathsf{A}^f\mathsf{B}^f &= (\mathsf{S}\mathsf{A}^e\mathsf{S}^{-1})(\mathsf{S}\mathsf{B}^e\mathsf{S}^{-1}) \\ &= \mathsf{S}\mathsf{A}^e(\mathsf{S}^{-1}\mathsf{S})\mathsf{B}^e\mathsf{S}^{-1} \\ &= \mathsf{S}\mathsf{A}^e(\mathsf{I})\mathsf{B}^e\mathsf{S}^{-1} \\ &= \mathsf{S}\mathsf{A}^e\mathsf{B}^e\mathsf{S}^{-1} \\ &= \mathsf{S}(\mathsf{A}^e\mathsf{B}^e)\mathsf{S}^{-1} \\ &= \mathsf{S}\mathsf{C}^e\mathsf{S}^{-1} \\ &= \mathsf{C}^f.\end{aligned}$$

Therefore, similarity preserves matrix multiplication. Suppose now that S is unitary, $\mathsf{S}^{-1} = \mathsf{S}^\dagger$, and that H^e is hermitian, $(\mathsf{H}^e)^\dagger = \mathsf{H}^e$. In the new basis the hermitian matrix is

$$\mathsf{H}^f = \mathsf{S}\mathsf{H}^e\mathsf{S}^{-1}.$$

Take the hermitian conjugate of both sides.

$$\begin{aligned}(\mathsf{H}^f)^\dagger &= (\mathsf{S}\mathsf{H}^e\mathsf{S}^{-1})^\dagger \\ &= [\mathsf{S}(\mathsf{H}^e\mathsf{S}^{-1})]^\dagger \\ &= (\mathsf{H}^e\mathsf{S}^{-1})^\dagger\mathsf{S}^\dagger \\ &= (\mathsf{S}^{-1})^\dagger(\mathsf{H}^e)^\dagger\mathsf{S}^\dagger \\ &= (\mathsf{S}^\dagger)^\dagger(\mathsf{H}^e)^\dagger\mathsf{S}^\dagger \\ &= \mathsf{S}\mathsf{H}^e\mathsf{S}^{-1} \\ &= \mathsf{H}^f\end{aligned}$$

Therefore, H^f is hermitian.

With respect to an orthonormal basis, the inner product of two vectors is $\langle\alpha\,|\,\beta\rangle = \mathsf{a}^\dagger\mathsf{b}$. The inner product with respect to the new basis is

$$\begin{aligned}
\langle\alpha\,|\,\beta\rangle^f &= (\mathsf{a}^f)^\dagger\mathsf{b}^f \\
&= (\mathsf{S}\mathsf{a}^e)^\dagger(\mathsf{S}\mathsf{b}^e) \\
&= (\mathsf{a}^e)^\dagger\mathsf{S}^\dagger(\mathsf{S}\mathsf{b}^e) \\
&= (\mathsf{a}^e)^\dagger(\mathsf{S}^\dagger\mathsf{S})\mathsf{b}^e \\
&\stackrel{?}{=} (\mathsf{a}^e)^\dagger(\mathsf{S}^{-1}\mathsf{S})\mathsf{b}^e \\
&= (\mathsf{a}^e)^\dagger(\mathsf{I})\mathsf{b}^e \\
&= (\mathsf{a}^e)^\dagger\mathsf{b}^e \\
&= \langle\alpha\,|\,\beta\rangle^e.
\end{aligned}$$

The inner products with respect to the new and old bases are equal if and only if $\mathsf{S}^{-1} = \mathsf{S}^\dagger$. Therefore, S carries an orthonormal basis into another orthonormal basis if and only if it is unitary.