Problem A.17

Prove that $\text{Tr}(\mathsf{T}_1\mathsf{T}_2) = \text{Tr}(\mathsf{T}_2\mathsf{T}_1)$. It follows immediately that $\text{Tr}(\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3) = \text{Tr}(\mathsf{T}_2\mathsf{T}_3\mathsf{T}_1)$, but is it the case that $\text{Tr}(\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3) = \text{Tr}(\mathsf{T}_2\mathsf{T}_1\mathsf{T}_3)$, in general? Prove it, or disprove it. *Hint:* The best disproof is always a counterexample—the simpler the better!

Solution

Consider element ij of the product of two $n \times n$ matrices, A and B.

$$(\mathsf{AB})_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Take the trace of AB.

$$\begin{aligned}
\text{Tr}(\mathsf{AB}) &= \sum_{i=1}^{n} (\mathsf{AB})_{ii} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ki}A_{ik} \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} B_{ki}A_{ik} \\
&= \sum_{k=1}^{n} (\mathsf{BA})_{kk} \\
&= \text{Tr}(\mathsf{BA})
\end{aligned}$$

Let C be a third $n \times n$ matrix. Because matrix multiplication is associative,

$$\text{Tr}[\mathsf{A}(\mathsf{BC})] = \text{Tr}[(\mathsf{BC})\mathsf{A}] \quad \text{and} \quad \text{Tr}[(\mathsf{AB})\mathsf{C}] = \text{Tr}[\mathsf{C}(\mathsf{AB})],$$

that is, $\text{Tr}(\mathsf{ABC}) = \text{Tr}(\mathsf{BCA}) = \text{Tr}(\mathsf{CAB})$. However, $\text{Tr}(\mathsf{ABC}) \neq \text{Tr}(\mathsf{BAC})$; for example, if

$$\mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$\mathsf{ABC} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{Tr}(\mathsf{ABC}) = 1 + 0 = 1$$

$$\mathsf{BAC} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{Tr}(\mathsf{BAC}) = 0 + 0 = 0.$$